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# FUNCTIONS
OF
# REAL VARIABLES

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BY

WILLIAM FOGG OSGOOD, PH.D., LL.D.

PROFESSOR OF MATHEMATICS, THE NATIONAL UNIVERSITY OF PEKING
PROFESSOR OF MATHEMATICS, HARVARD UNIVERSITY, EMERITUS

HAFNER PUBLISHING COMPANY, INC.
NEW YORK

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Reprint 1938
Reprint 1947

# PREFACE

The book is an elaboration of lectures given at The National University of Peking in the fall and winter of 1934-35. It is intended for students who have completed a course in Advanced Calculus, but have not as yet entered on the study of the Theory of Functions, either real or complex. Emphasis is laid on the new methods, and these are illustrated by many and varied applications, and numerous exercises.

Chapter I deals with the convergence of infinite series. The student has hitherto been concerned chiefly with the results of limiting processes, the applications of the Calculus. As the next step he needs training in the more abstract use of simple limits as applied to problems of a purely analytic nature. He needs to live in this new domain of thought, in which the limiting process is the central idea. And he needs to work many simple problems, in which the existence of the limit is the chief end.

Chapter II. The Number System. The student has thus far taken the system of real numbers for granted, and worked with them. He may continue to do so to the end of his life without detriment to his mathematical thought. Thus he may omit this chapter altogether, taking the Theorem of Continuity, § 5, as a postulate, but noting, of course, the definitions and theorems of §§ 6, 8, 10. On the other hand, most mathematicians are curious, at one time or other in their lives, to see how the system of real numbers can be evolved from the natural numbers. Dedekind's answer is not the only one, or even the simplest one for the beginner; but it is one no mathematician can ignore and to which, probably, most mathematicians will give first place.

With Chapter III, Point Sets, begin the new concepts which are fundamental in the Theory of Functions of Real Variables, and this chapter may form the starting point for the beginner, so far as any logical difficulties go. The concept: *function*, is treated in §§ 3, 4, and now comes the definition of *limit*, § 5. The proofs chosen for the Three Theorems on Continuous Functions may well be criticised.

Should not the writer pick out the method most easily accessible for the student (say, the method of nested intervals) and use that one method systematically? This is precisely what the student should do, using these other proofs as exercises. Let me say right here that the only way in which the student can hope to attain mastery of the subject, is to write his own book. He should take each theorem by itself, state it in his own language, and prove it as the author *ought* to have proved it for that student's needs. The clearer the presentation in a text-book is, the worse for the student who would rely on *reading*. The student *must himself* produce his own independent proofs of the main theorems, in writing, and then come back again and again to his own presentation, as he walks to a lecture or strolls through his favorite haunts in this enchanting city. First, the *content* of the theorem; secondly, the *method of proof*, must go over into his flesh and blood. It is a question of *habits of thought*, and habits are formed by repetition.

To come back to the three theorems on continuity, the student should ultimately dominate all the methods set forth. But it is enough at the beginning to have one method of proof for each theorem, and the choice is left to him. Let it be said that no one can *teach* the student the Theory of Functions. For the Theory of Functions is a habit of thought, not a set of rules to be applied like the formulas of differentiation.

In saying these things we have really forecast the whole study of the uniform question. The difficulty with uniform continuity and the uniform convergence of series and integrals lies not so much in the definitions, and in the statement and proof of the theorems, as in recognizing a double-limit question when it arises in practice. For these questions are not labelled:—"Here is a double-limit." It is on this account that especial pains have been taken both to illustrate the situation geometrically by graphs, and analytically to formulate the situation as a

$$\lim_{n=\infty}\left[\lim_{m=\infty} s(n, m)\right] = \lim_{m=\infty}\left[\lim_{n=\infty} s(n, m)\right]$$

question. It is with deep regret that the writer feels forced to treat systematically the double-limit questions for proper integrals. The student, after mastering Chapter V on Uniform Convergence of

Infinite Series, should find his own questions for the proper definite integral:

$$\int_a^b f(x, \alpha)\, dx,$$

and treat them independently. He can still do this, in the main, by reading only casually what is said in the text of Chapter IX and then at once setting about his own individual elaboration of this whole subject. Even the case of improper integrals:

$$\int_c^\infty f(x, \alpha)\, dx,$$

can be treated in like manner, and it is not till the question of the reversal of order of integration in the iterated improper integral

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy$$

arises, that he needs specific guidance. Here, again, however is an opportunity for independent thought. Let him do Chapter IX with a minimum amount of help from the book.

But method is not the sole topic, nor should it appear to the student as an end in itself. His taste in analysis must be cultivated, and to this end, what better material than the elementary functions, developed out of their differential equations and leading to their expansion into series of fractions and infinite products; the $\Gamma$- and $B$-functions with their integrals and products, and a first suggestion of asymptotic expansions; Fourier's series and the development problem, with an example of a divergent series, uniformly summable; Fourier's integral, with applications, and extension to functions of several variables; finally, differential equations, total and partial.

This last subject has the widest ramifications in pure and applied mathematics, and courses of the nature of rule of thumb are everywhere current. Two things the student needs to know in advance, namely: the intrinsic meaning of a differential equation, and the existence theorems governing the solution in a restricted region. The proof of the latter affords opportunity for expounding, on the hand of this important application, the greatest single method in all science — the Method of Successive Approximations.

The plan of publishing these Lectures was first suggested to me by my old friend and pupil, and present colleague at Peita, Professor Kiang Tsai-Han, who has accompanied their preparation with warm interest and generous support.

It is a pleasure to express my deep appreciation of the efficient cooperation of the University Press, The National University of Peking, in this difficult task of mathematical composition. My former Assistant, Mr. Hsü Pao-lu, was most helpful in preparing the type-written copy for press. My present Assistant, Mr. Sun Shu-Peng, has been of inestimable service to me in seeing the book through the press. His keen interest in securing accuracy of detail, and his excellent common sense, have made him a most extraordinary helper in this undertaking.

PEIPING
December 17, 1935

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# FUNCTIONS OF REAL VARIABLES

## Chapter I

## Convergence of Infinite Series

**§ 1. Definitions.** Let $u_0, u_1, u_2, \cdots$ be any set of numbers proceeding according to a definite law. Form the sum:

$$s_n = u_0 + u_1 + \cdots + u_{n-1},$$

and allow $n$ to increase without limit. If $s_n$ approaches a limit, denote the latter by $U$:

$$\lim_{n=\infty} s_n = U.$$

The expression

$$1) \qquad u_0 + u_1 + \cdots$$

is called an *infinite series* (or, more simply, a *series*). It is said to be *convergent* if $s_n$ approaches a limit, $U$; and this number $U$ is assigned to it as its *value.* $U$ is sometimes called the *sum*, but this nomenclature is unfortunate, since it is impossible to add an infinite number of terms together.

If $s_n$ approaches no limit, the series is said to be *divergent*, and no number is assigned to it as a value.*

A familiar example of a convergent series is the geometric progression

$$2) \qquad a + ar + ar^2 + \cdots,$$

where $r$ is numerically less than unity. Here,

$$3) \qquad s_n = a + ar + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}$$

and

$$\lim_{n=\infty} s_n = \frac{a}{1-r}.$$

Hence

$$4) \qquad \frac{a}{1-r} = a + ar + ar^2 + \cdots, \qquad |r| < 1.$$

---

* To certain divergent series values are attached through other convergent processes, some of which will be considered later.

A further example is the series

$$5)\qquad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots,$$

whose value is 1. Here,

$$s_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

or

$$s_n=1-\frac{1}{n+1},\qquad \lim_{n=\infty} s_n=1.$$

The early mathematicians thought that a series must converge if the general term approaches 0 as its limit: $\lim\limits_{n=\infty} u_n=0$. But this is not the case. As an example, consider the *harmonic series*,

$$6)\qquad 1+\frac{1}{2}+\frac{1}{3}+\cdots.$$

It is possible to find $n$ terms, the sum of which will exceed any given number, no matter how large. For, obviously,

$$\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{m+m}>\frac{1}{2}$$

since these terms (except the last) exceed respectively the terms

$$\frac{1}{m+m}+\frac{1}{m+m}+\cdots+\frac{1}{m+m}\quad \text{(to } m \text{ terms)},$$

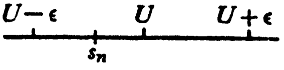
and the value of this sum is ½. So we see that, striking in anywhere in the series, we can add a definite number of terms which will yield a sum greater than ½, and hence $s_n$ increases without limit as $n$ increases. Thus the harmonic series diverges.

THEOREM. *A necessary condition for the convergence of an infinite series is, that the general term approach* 0 *as its limit.*

To deny the truth of the theorem is to assert that there exists a positive constant, $h$, such that, no matter how large $m$ be chosen, there will always be a term, $u_p$, for which

$$|u_p|>h,\qquad p>m.$$

But here is a contradiction. Mark off an interval about the point $U$, extending from $U-\epsilon$ to $U+\epsilon$. Since $s_n$ approaches $U$ as its limit, the points $s_n$ will all remain within this interval when $n$



is greater than a suitably chosen integer, $\mu$:

$$U - \epsilon < s_n < U + \epsilon, \qquad \mu \leqq n.$$

Here, $\epsilon$ may be any positive quantity. Choose $\epsilon = \frac{1}{2} h$. There will be a term $u_p$ such that

$$|u_p| > h, \qquad p > \mu.$$

Now, $s_{p-1}$ lies in the interval $(U - \epsilon, U + \epsilon)$. But, since

$$s_p = s_{p-1} + u_p,$$

$s_p$ lies outside the interval. From this contradiction follows the truth of the theorem.

EXERCISES

1. If the series 1) converges, show that the series

7) $$u_m + u_{m+1} + \cdots$$

converges, where $m$ is any fixed integer.

2. If the series 7) converges, show that the series 1) converges.

3. If the series 1) converges, show that the series

8) $$k u_0 + k u_1 + \cdots,$$

where $k$ is any fixed number, converges.

4. Is the converse of the theorem of Question 3 true? Prove your answer to be correct.

## §2. Comparison Tests for Series of Positive Terms.

Series whose terms are all positive or 0 can be tested for convergence by the following theorem.

DIRECT COMPARISON TEST FOR CONVERGENCE *Let*

$$u_0 + u_1 + \cdots, \qquad 0 \leqq u_n,$$

*be a series of positive (or zero) terms to be tested for convergence. Let*

$$a_0 + a_1 + \cdots, \qquad 0 \leqq a_n,$$

*be a series known to converge, and let*

$$u_n \leqq a_n, \qquad m \leqq n.$$

*Then the given series converges.*

Consider the sums:

$$s_n = u_0 + u_1 + \cdots + u_{n-1},$$
$$S_n = a_0 + a_1 + \cdots + a_{n-1}.$$

If $m = 0$, we have:

$$s_n \leqq S_n.$$

Let $A$ be the value of the $a$-series. Then

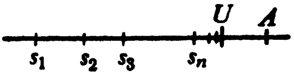
$$S_n \leqq A.$$

Hence

$$s_n \leqq A.$$

Thus we have in $s_n$ a variable which always increases (or remains constant) as $n$ increases, but which never exceeds a fixed number, $A$. Such a variable always approaches a limit, $U$, which in general is less than $A$, but may, in particular, $= A$.

The truth of this statement is plausible if we think of $s_n$ as represented by a point on a line — the scale of numbers. For, these points move to the right — if they move at all — as $n$ increases. But they never pass beyond the point $A$. They must, therefore, condense on some point, $U$, to the left of $A$, or possibly on $A$ itself.



An arithmetic proof of this basal principle will be given later. At present, we accept it as granted, and formulate it as a

FUNDAMENTAL PRINCIPLE. *If $s_n$ is a variable which always increases or remains the same as $n$ increases, but never exceeds some fixed number, $A$:*

i) $$s_n \leqq s_{n+1};$$

ii) $$s_n \leqq A,$$

*then $s_n$ approaches a limit, $U$:*

$$\lim_{n=\infty} s_n = U.$$

*Moreover,* $U \leqq A$.

To complete the proof of the Test for Convergence it remains to remove the restriction that $m = 0$. For an unrestricted $m$, observe that

$$s_n = u_0 + u_1 + \cdots + u_{m-1} + u_m + u_{m+1} + \cdots + u_{n-1}, \qquad m < n.$$

The variable

$$u_m + u_{m+1} + \cdots + u_{m-k+1}$$

approaches a limit when $k$ increases indefinitely, as has already been proved. But $s_n$ differs from this variable merely by an additive constant. Hence $s_n$ approaches a limit, and the proof is complete.

We have made use of the fact that if each of two variables approaches a limit, their sum approaches a limit. The truth of this theorem is obvious geometrically. An arithmetic proof will be given latter.

*Example.* Consider the series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

Compare its terms, beginning with the third, with those of the convergent geometric series,

$$\frac{1}{2} + \frac{1}{2^2} + \cdots.$$

Since

$$1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n + 1) > 2 \cdot 2 \cdot \cdots \cdot 2 \text{ (to } n \text{ factors)},$$

it follows that

$$\frac{1}{(n + 1)!} < \frac{1}{2^n},$$

and hence the given series converges.

Direct Comparison Test for Divergence. *Let*

$$a_0 + a_1 + \cdots, \qquad 0 \leqq a_n, \qquad m \leqq n,$$

*be a divergent series, and let*

$$a_n \leqq u_n, \qquad m \leqq n.$$

*Then the series*

$$u_0 + u_1 + \cdots$$

*diverges.*

The proof is left to the reader.

### Exercises

Test the following series for convergence or divergence.

1. $1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$

2. $1 + \frac{1}{3} + \frac{1}{5} + \cdots$

3. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

4. $$ar + ar^4 + ar^9 + \cdots + ar^{n^2} + \cdots, \qquad 0 \leqq r.$$

5. $$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \qquad 0 \leqq x.$$

6. $$x^2 + \frac{x^6}{3} + \frac{x^{10}}{5} + \cdots$$

7. $$\frac{x^2}{1 + x^2} + \frac{x^4}{1 + x^4} + \frac{x^6}{1 + x^6} + \cdots$$

8. Prove the theorem:— Given the series

$$u_0 + u_1 + \cdots, \qquad 0 \leqq u_n, \qquad m \leqq n.$$

If $n\,u_n$ approaches a limit as $n$ becomes infinite, and if this limit is positive, the series diverges.

9. Show that the series

$$\frac{1}{1 + x} + \frac{1}{2 + x} + \cdots$$

diverges for all values of $x$ for which it is defined:

$$-\infty < x < +\infty, \qquad x \neq -1, -2, \cdots.$$

10. Given the series:

$$u_0 + u_1 + \cdots, \qquad 0 \leqq u_n, \qquad m \leqq n.$$

If $\sqrt[n]{u_n}$ approaches a limit as $n$ becomes infinite, and if this limit is less than 1, the series converges:

$$\lim_{n=\infty} \sqrt[n]{u_n} = \gamma < 1.$$

But if $\gamma > 1$, the series diverges.

11. Given a power series:

$$a_0 + a_1 x + a_2 x^2 + \cdots, \qquad 0 < a_n, \qquad m \leqq n.$$

Let

$$\lim_{n=\infty} \sqrt[n]{a_n} = \rho > 0.$$

Then the series converges for all values of $x$ such that $0 \leqq x < 1/\rho$.

What if $\sqrt[n]{a_n}$ becomes infinite?

12. Given a series:

$$u_0 + u_1 + \cdots, \qquad 0 < u_n, \qquad m \leqq n.$$

If

$$\log \frac{u_n}{u_{n+1}} > \gamma, \qquad m \leqq \mu \leqq n,$$

where $\gamma$ is a positive constant, the series converges.

## § 3. A General Test-Ratio Test.

*Let*

$$a_0 + a_1 + \cdots, \qquad 0 < a_n, \qquad m \leqq n,$$

*be a convergent series, and let*

$$u_0 + u_1 + \cdots, \qquad 0 < u_n, \qquad m \leqq n,$$

*be a series to be tested for convergence. If*

$$\frac{u_{n+1}}{u_n} \leqq \frac{a_{n+1}}{a_n}, \qquad m \leqq n,$$

*the u-series converges.*

*Proof.* By hypothesis,

$$\frac{u_{m+1}}{u_m} \leqq \frac{a_{m+1}}{a_m},$$

$$\frac{u_{m+2}}{u_{m+1}} \leqq \frac{a_{m+2}}{a_{m+1}},$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\frac{u_{m+k}}{u_{m+k-1}} \leqq \frac{a_{m+k}}{a_{m+k-1}}.$$

Multiply the $k$ inequalities together. Thus

$$\frac{u_{m+k}}{u_m} \leqq \frac{a_{m+k}}{a_m},$$

$$u_n \leqq \frac{u_m}{a_m} a_n.$$

The factor $u_m/a_m$ is a constant. Hence the series whose general term is $(u_m/a_m)\, a_n$ converges, and consequently the $u$-series converges.

In the proof, we have made use of the theorem that if each of two variables approaches a limit, their product approaches a limit. The proof of this theorem will be given later.

**Divergence Test.** *Let*

$$a_0 + a_1 + \cdots, \qquad 0 < a_n, \qquad m \leqq n,$$

*be a divergent series, and let*

$$u_0 + u_1 + \cdots, \qquad 0 < u_n, \qquad m \leqq n,$$

*be a series to be tested for divergence. If*

$$\frac{a_{n+1}}{a_n} \leqq \frac{u_{n+1}}{u_n}, \qquad m \leqq n,$$

*the u-series diverges.*

The proof is similar to the foregoing, and can be left to the reader.

To illustrate these theorems, let the *a*-series be the geometric series

$$a + ar + ar^2 + \cdots, \qquad 0 < a, \qquad 0 < r < 1.$$

Then

$$\frac{a_{n+1}}{a_n} = r.$$

We are thus led to the familiar

First Test-Ratio Test. *If*

$$u_0 + u_1 + \cdots .$$

*be a series whose terms from a definite point on are all positive, and if*

$$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = \mathbf{r} < 1,$$

*the series converges. But if* $\mathbf{r} > 1$, *the series diverges. If* $\mathbf{r} = 1$, *there is no test.*

For, we can choose $r$ between $\mathbf{r}$ and unity, $\mathbf{r} < r < 1$, or $1 < r < \mathbf{r}$, and then the hypotheses of the general theorem will be fulfilled.

If $\mathbf{r} = 1$, the *u*-series may converge and it may diverge, as is seen from the two examples:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \qquad \text{(convergent)}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots \qquad \text{(divergent)}$$

### EXERCISES

1. Show that, if

$$\lim_{n=\infty} \log \frac{u_n}{u_{n+1}} = \sigma, \qquad (0 < u_n, \quad m \leqq n)$$

the series converges when $\sigma > 0$, and diverges when $\sigma < 0$.

2. Show that if

$$1 \leqq \frac{u_{n+1}}{u_n}, \qquad (0 < u_n, \quad m \leqq n)$$

the series diverges.

**§ 4. Cauchy's Test by Integration.** It happens in simple important cases that the function $u_n$ of the discrete variable $n$ admits useful interpolation for a continuous argument $x$. Thus

$$\frac{1}{n}, \qquad \frac{1}{n^p}, \qquad \frac{1}{n \log n}, \qquad \text{etc.}$$

are interpolated by the continuous functions

$$\frac{1}{x}, \qquad \frac{1}{x^p}, \qquad \frac{1}{x \log x}, \qquad \text{etc.}$$

Cauchy devised a simple method for determining the convergence of such series.

CONVERGENCE TEST. *Let* $f(x)$ *be a positive, continuous, decreasing, monotonic function,*

$$f(x') \geqq f(x'') \geqq 0, \qquad x' < x''.$$

*Form the integral*

$$\int_g^\infty f(x)\, dx.$$

*If this integral converges, the series*

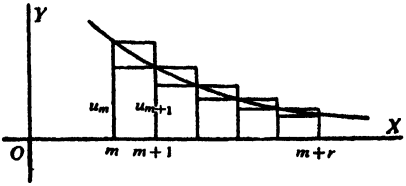
$$u_0 + u_1 + \cdots$$

*converges. If the integral diverges, the series diverges.*

The proof follows immediately from inspection of the figure, p. 10. For obviously

$$u_{m+1} + u_{m+2} + \cdots + u_{m+r} \leqq \int_m^{m+r} f(x)\, dx \leqq \int_m^\infty f(x)\, dx.$$

When this last integral converges, the sum satisfies the hypotheses of the Fundamental Principle of § 2, $A$ being the value of the integral, and so the series converges.



On the other hand,

$$\int_m^{m+r} f(x)\,dx \leqq u_m + u_{m+1} + \cdots + u_{m+r-1}.$$

If, then, the integral diverges, the series diverges.

*Example.* Consider the harmonic series,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Here,

$$f(x) = 1/x,$$

$$\int_m^x \frac{dx}{x} = \log x - \log m.$$

Hence the integral in question diverges and the series diverges.

On the other hand the series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots, \qquad 1 < p,$$

leads to the function $1/x^p$ and to the integral

$$\int_m^x \frac{dx}{x^p} = \frac{1}{p-1}\left[\frac{1}{m^{p-1}} - \frac{1}{x^{p-1}}\right]$$

The crucial integral is thus seen to converge, and so the series converges.

*Appraisal of the Error.* Cauchy was much interested in appraising the error made in computing the value of a series, when one breaks off with the $m$-th term. Let

$$U = u_1 + u_2 + \cdots = s_n + r_n,$$

$$r_n = u_{n+1} + u_{n+2} + \cdots,$$

$$A = \int_m^\infty f(x)\,dx.$$

Then

$$r_m \leqq A \leqq r_{m-1}.$$

But

$$r_{m-1} - r_m = u_m.$$

Hence we obtain the following formula for the appraisal of the error:

$$r_m = A - \theta u_m, \qquad 0 < \theta < 1.$$

Thus

$$U = s_m + A - \theta u_m,$$

and we may stop adding terms as soon as we come to one which is less than the limit of error assigned.

Consider, in particular, the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad 1 < p.$$

Here,

$$r_m = \frac{1}{(m+1)^p} + \frac{1}{(m+2)^p} + \cdots$$

$$A = \int_m^{\infty} \frac{dx}{x^p} = \frac{1}{p-1} \frac{1}{m^{p-1}}.$$

Thus the error made by breaking off with the $m$-th term:

$$s_m = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{m^p},$$

is given by the equation:

$$r_m = \frac{1}{p-1} \frac{1}{m^{p-1}} - \theta \frac{1}{m^p},$$

and

$$U = 1 + \frac{1}{2^p} + \cdots + \frac{1}{m^p} + \frac{1}{p-1} \cdot \frac{1}{m^{p-1}} - \frac{\theta}{m^p}$$

$$= 1 + \frac{1}{2^p} + \cdots + \frac{1}{(m-1)^p} + \frac{1}{p-1} \frac{1}{m^{p-1}} + \frac{\theta'}{m^p},$$

$$0 < \theta' < 1.$$

Let it be required to compute the value of this series for $p = 1.01$ correct to $\frac{1}{10}$. Here the first term less than $\frac{1}{10}$ is $10^{-p}$. Hence

$$s_{m-1} = s_9 = 1 + \frac{1}{2^{1.01}} + \cdots + \frac{1}{9^{1.01}}.$$

$A$ has the value:

$$A = \frac{1}{0.01} \cdot \frac{1}{10^{0.01}} = 97.72,$$

and it remains to compute the sum $s_9$.

## EXERCISES

1. Show that the exact computation of the term $u_m$ can be avoided by using the formula:

$$r_{m-1} = A + \theta u_m, \qquad 0 < \theta < 1,$$

where $A$ has the same value as before, but $\theta$ is different. Thus

$$U = s_{m-1} + A + \theta u_m.$$

2. Study the series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

when $p = 1.001$. How many terms must be taken, in order that the error made by dropping the remainder may be less than 1? How many, if the approximate formula for the remainder be used? Compute the value of the series correct to the nearest integer.

3. Show that the series

$$\frac{1}{2 \log 2} + \frac{1}{3 \log 3} + \cdots$$

diverges.

4. Show that the series

$$\frac{1}{2 (\log 2)^p} + \frac{1}{3 (\log 3)^p} + \cdots, \qquad 1 < p,$$

converges.

5. Treat the series

$$\sum \frac{1}{n \log n (\log \log n)^p},$$

and generalize.

6. Apply Cauchy's test to the geometric series,

$$a + ar + ar^2 + \cdots, \qquad 0 < a, \qquad 0 < r.$$

7. Prove the theorem:— Given the series:

$$u_0 + u_1 + \cdots, \qquad 0 \leqq u_n, \qquad m \leqq n.$$

If

$$\lim_{n=\infty} n^p u_n = 0, \qquad 1 < p,$$

the series converges. Here, $p$ is a constant.

8. Show that the series

$$\frac{1}{1+x} + \frac{1}{2^p + x} + \frac{1}{3^p + x} + \cdots, \qquad 1 < p,$$

converges for all values of $x$ for which its terms are defined:

$$-\infty < x < +\infty, \qquad x \neq -1, -2^p, -3^p, \cdots$$

## § 5. Tests in Case $\lim u_{n+1}/u_n = 1$.

When

$$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = 1,$$

the First Test-Ratio Test fails. For example, if

$$u_n = \frac{1}{n^p}, \qquad \lim_{n=\infty} \frac{u_{n+1}}{u_n} = 1,$$

regardless of the value of $p$. Now, if $p$ is very large, $10!^{10!}$ say, the early terms of the series drop off rapidly. But the convergence of this series cannot be established by comparison with a geometric series,

$$\frac{1}{n^p} < ar^n.$$

For, no matter how large $a$ be taken, and no matter how near to unity $r < 1$ may be chosen, the inequality will ultimately point the other way, since

$$\lim_{x=\infty} x^p r^x = 0.$$

It is easy, however, to develop limit tests which will apply to series which converge or diverge commensurately with the series

$$\sum \frac{1}{n^p}.$$

Here,

$$\frac{a_{n+1}}{a_n} = \frac{n^p}{(n+1)^p} = \left(1 + \frac{1}{n}\right)^{-p} = 1 - \frac{p}{n} + r_n,$$

where $r_n$ is of the order $1/n^2$.
Hence,[*]

$$\lim_{n=\infty} n\left(1-\frac{a_{n+1}}{a_n}\right)=p.$$

Consider now a series such that

$$\lim_{n=\infty} n\left(1-\frac{u_{n+1}}{u_n}\right)=\rho.$$

First, let $\rho > 1$. Choose $1 < p < \rho$. Then, for large values of $n$, the variable

$$n\left(1-\frac{a_{n+1}}{a_n}\right) \quad \text{will be near } p$$

and

$$n\left(1-\frac{u_{n+1}}{u_n}\right) \quad \text{will be near } \rho.$$

Hence

$$n\left(1-\frac{a_{n+1}}{a_n}\right) < n\left(1-\frac{u_{n+1}}{u_n}\right), \qquad m \leqq n,$$

and thus

$$\frac{u_{n+1}}{u_n} < \frac{a_{n+1}}{a_n}, \qquad m \leqq n.$$

We see, then, that the $u$-series converges. Similarly, if $p < 1$, it appears that the $u$-series diverges. We are thus led to the following

SECOND TEST-RATIO TEST. *Let*

$$u_0 + u_1 + \cdots, \qquad 0 < u_n, \qquad m \leqq n,$$

---

[*] It is not necessary to base this result on the binomial series, or even Taylor's Theorem with the Remainder (carried to the quadratic term). Write

$$n\left(1-\frac{a_{n+1}}{a_n}\right) = -\frac{\left(1+\frac{1}{n}\right)^{-p}-1}{\frac{1}{n}}.$$

Now, the limit

$$\lim_{x=0}\frac{(1+x)^{-p}-1}{x}$$

is shown at once by the Calculus to be $-p$, and we are through.

*be a series such that*

$$\lim_{n=\infty} n\left(1-\frac{u_{n+1}}{u_n}\right)=\rho \quad \text{or} + \infty \text{ or } - \infty.$$

*If* $\rho > 1$ *or* $+\infty$, *the series converges. If* $\rho < 1$ *or* $-\infty$, *the series diverges. If* $\rho = 1$, *there is no test.*

The test for divergence can be made somewhat sharper. For, if

$$n\left(1-\frac{u_{n+1}}{u_n}\right) \leqq 1, \qquad m \leqq n,$$

then

$$\frac{n-1}{n} \leqq \frac{u_{n+1}}{u_n}.$$

On the left stands the test-ratio of the harmonic series. Hence the $u$-series diverges.

### EXERCISES

In the following exercises,

$$u_0 + u_1 + \cdots$$

shall be a series whose terms ultimately become and remain positive. (It would do as well, of course, if the terms were ultimately to become and remain negative.)

1. If

$$\lim_{n=\infty}\left[n-(n+1)\frac{u_{n+1}}{u_n}\right]=\sigma \text{ or } + \infty \text{ or } - \infty,$$

show that the series

i) converges when $\sigma > 0$ or $+\infty$;

ii) diverges when $\sigma < 0$ or $-\infty$.

2. If

$$\lim_{n=\infty} n \log \frac{u_n}{u_{n+1}} = \sigma \text{ or } + \infty \text{ or } - \infty,$$

show that the series

i) converges when $\sigma > 0$ or $+\infty$;

ii) diverges when $\sigma < 0$ or $-\infty$.

3. If the test-ratio is of the form:

$$\frac{u_{n+1}}{u_n} = 1 - \frac{c}{n} + r_n,$$

where

$$\lim_{n=\infty} n r_n = 0,$$

and $c$ is a constant, show that the series will converge if $c > 1$ and diverge if $c < 1$.

**§6. Kummer's Criterion.** One of the earliest attempts to obtain general criteria for convergence and divergence is found in a paper by Kummer* of the year 1835. Let

$$1) \qquad u_0 + u_1 + \cdots, \qquad 0 < u_n, \qquad m \leqq n,$$

be a series to be tested for convergence. Let $P_m$, $P_{m+1}$, $\cdots$ and $a$ be any set of positive numbers. Form the expression:

$$2) \qquad \omega_n = \frac{P_n u_n}{a} - \left(\frac{P_{n+1}}{a} + 1\right) u_{n+1}.$$

On writing this equation for $n = m,\ m + 1, \cdots, m + r - 1$ and adding, we find:

$$3) \qquad \omega_m + \omega_{m+1} + \cdots + \omega_{m+r-1} =$$
$$\frac{P_m u_m}{a} - \frac{P_{m+r} u_{m+r}}{a} - u_{m+1} - u_{m+2} - \cdots - u_{m+r}.$$

Hence

$$4) \qquad u_{m+1} + u_{m+2} + \cdots + u_{m+r} =$$
$$\frac{P_m u_m}{a} - \frac{P_{m+r} u_{m+r}}{a} - \omega_m - \omega_{m+1} - \cdots - \omega_{m+r-1}.$$

LEMMA. *If positive numbers $P_m$, $P_{m+1}$, $\cdots$ and a positive number $a$, can be found, such that $\omega_n \geqq 0$, $m \leqq n$; i.e. if*

$$5) \qquad 0 \leqq \frac{P_n u_n}{a} - \left(\frac{P_{n+1}}{a} + 1\right) u_{n+1}, \qquad 0 < u_n, \qquad m \leqq n.$$

*then the u-series converges.*

For, from 4) we infer that

$$u_{m+1} + \cdots + u_{m+r} < \frac{P_m u_m}{a},$$

and since the right-hand member is constant, the theorem is proved.

The condition 5) is equivalent to the following:

$$6) \qquad 0 < a \leqq P_n \frac{u_n}{u_{n+1}} - P_{n+1}.$$

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* *Journal für Mathematik*, vol. 13 (1835) p. 171.

This condition will surely be fulfilled whenever

$$P_n \frac{u_n}{u_{n+1}} - P_{n+1}$$

approaches a limit and this limit is positive:

$$\lim_{n=\infty} \left[ P_n \frac{u_n}{u_{n+1}} - P_{n+1} \right] = \sigma > 0.$$

For, it will then be possible to choose an $a$ between 0 and $\sigma$, and so, from a definite point $n = m' \geqq m$ on, we shall have 6) holding.

We are thus led to the following test for convergence, which however, without any restrictive hypothesis relating to the signs of the terms, applies to any real series.

TEST FOR CONVERGENCE. *Let*

$$u_0 + u_1 + \cdots$$

*be any real series. If positive numbers* $P_m$, $P_{m+1}, \cdots$ *can be found such that the variable*

$$P_n \frac{u_n}{u_{n+1}} - P_{n+1}$$

*approaches a positive limit* $\sigma$, *or becomes positively infinite:*

7) $$\lim_{n=\infty} \left[ P_n \frac{u_n}{u_{n+1}} - P_{n+1} \right] = \sigma > 0, \quad \text{or} = +\infty,$$

*then the u-series converges.*

For it is obvious that $u_n/u_{n+1}$ must ultimately become and remain positive, and so the series, from a definite point on, must have its terms all of the same sign.

TEST FOR DIVERGENCE. *Let*

$$u_1 + u_2 + \cdots, \qquad 0 < u_n, \qquad m \leqq n,$$

*be a given series. Let* $P_m$, $P_{m+1}, \cdots$ *be a set of positive numbers such that the series*

8) $$\frac{1}{P_m} + \frac{1}{P_{m+1}} + \cdots$$

*diverges. If the variable*

$$P_n \frac{u_n}{u_{n+1}} - P_{n+1}$$

*approaches a negative limit $\sigma$ or becomes negatively infinite:*

$$9) \qquad \lim_{n=\infty}\left\{P_n\frac{u_n}{u_{n+1}} - P_{n+1}\right\} = \sigma < 0, \quad \text{or} = -\infty,$$

*then the u-series diverges.*

*Proof.* From a definite point on, $n \geqq \mu \geqq m$, we shall have:

$$P_n\frac{u_n}{u_{n+1}} - P_{n+1} < 0.$$

Hence

$$\frac{P_n}{P_{n+1}} < \frac{u_{n+1}}{u_n}, \qquad \mu \leqq n.$$

But the left-hand side is the test-ratio of a divergent series, 8). This proves the theorem.

A sufficient condition for divergence is, that

$$P_n u_n - P_{n+1} u_{n+1} \leqq 0,$$

provided the series 8) diverges.

A set of useful choices for $P_n$ is the following:

1. $P_n = 1$. This gives the First Test-Ratio Test.
2. $P_n = n$. This gives the Second Test-Ratio Test.
3. $P_n = n \log n$. This given the Third Test-Ratio Test. etc.

### EXERCISES

1. Assuming that the series

$$\sum\frac{1}{n}, \qquad \sum\frac{1}{n\log n}, \qquad \sum\frac{1}{n\log n\log\log n}, \qquad \text{etc.}$$

have been shown to be divergent by Cauchy's test, § 4—or otherwise—apply the Second, Third, etc. Tests for Convergence and Divergence to the series

$$\sum\frac{1}{n^p}, \qquad \sum\frac{1}{n(\log n)^p}, \qquad \sum\frac{1}{n\log n(\log\log n)^p}, \cdots$$

**§ 7. Continuation. Discussion.** Given any convergent series 1), it is always possible to find a set of positive $P_n$'s and a positive $a$ such that $\omega_n \geqq 0$. For 3) is equivalent to the relation

$$P_{m+r}\, u_{m+r} = P_m\, u_m - a(\omega_m + \omega_{m+1} + \cdots + \omega_{m+r-1} + u_{m+1} + u_{m+2} + \cdots + u_{m+r}).$$

If, then, $\omega_n \geqq 0$ is the general term of a convergent series, $a > 0$ can then be so chosen that the right-hand side of the equation will always be positive, and $P_{m+r}$ can then be determined from this equation ($P_m > 0$, arbitrary at the start).

Moreover, the $P_n$ and $a$ can be so determined that

$$P_n \frac{u_n}{u_{n+1}} - P_{n+1}$$

will approach any positive limit, $\sigma$. For, let

$$\omega_n = u_{n+1},$$

and determine the $P_n$ accordingly. Then 2) gives

$$P_n \frac{u_n}{u_{n+1}} - P_{n+1} = 2a,$$

and it is sufficient to set $2a = \sigma$.

Secondly, given any divergent series 1), a set of positive $P_n$'s can be found such that the series

$$\sum_{n=0}^{\infty} \frac{1}{P_n}$$

diverges and the variable

$$P_n \frac{u_n}{u_{n+1}} - P_{n+1}$$

approaches an arbitrary negative limit, $\sigma$.

In fact, we need only determine the $P_n$'s from the relation:

$$P_n \frac{u_n}{u_{n+1}} - P_{n+1} = \sigma < 0.$$

Here,

$$P_{n+1}\, u_{n+1} = P_n\, u_n + (-\sigma)\, u_{n+1}.$$

From this equation follows:

$$P_{m+r}\, u_{m+r} = P_m\, u_m + (-\sigma)(u_{m+1} + \cdots + u_{m+r}).$$

Thus a set of positive $P_n$'s is determined. Will the series

$$\sum \frac{1}{P_n}$$

diverge? It will. For, strike in anywhere and add terms:

$$\frac{1}{P_{\mu+1}}+\frac{1}{P_{\mu+2}}+\cdots+\frac{1}{P_{\mu+\rho}}.$$

Let

$$A=P_m u_m+(-\sigma)(u_{m+1}+\cdots+u_\mu).$$

Then

$$\frac{1}{P_{\mu+k}}=\frac{u_{\mu+k}}{A+(-\sigma)(u_{\mu+1}+\cdots+u_{\mu+k})},$$

$$\frac{1}{P_{\mu+1}}+\frac{1}{P_{\mu+2}}+\cdots+\frac{1}{P_{\mu+\rho}}>$$

$$\frac{u_{\mu+1}+u_{\mu+2}+\cdots+u_{\mu+\rho}}{A+(-\sigma)(u_{\mu+1}+u_{\mu+2}+\cdots+u_{\mu+\rho})}.$$

By taking $\rho$ large enough we can bring this last fraction as near to $1/(-\sigma)$ as we please, and so, in particular, make

$$\frac{1}{P_{\mu+1}}+\frac{1}{P_{\mu+2}}+\cdots+\frac{1}{P_{\mu+\rho}}>\frac{1}{2(-\sigma)}.$$

Hence the series $\Sigma\, 1/P_n$ diverges, and this completes the proof.

To sum up, then: An arbitrary series, $0<u_n$, $m\leqq n$, can be proved convergent by the Convergence Test, or divergent by the Divergence Test, if the $P_n$'s be suitably chosen.

Thirdly, for the divergence test, it is not enough to demand merely that a set of positive $P_n$'s can be found such that

$$P_n\frac{u_n}{u_{n+1}}-P_{n+1}$$

approaches a negative limit. For, if the $u$-series be a convergent series of postive terms and we determine $P_n$'s as in the last case, these $P_n$'s will all be positive. But the series $\Sigma\, 1/P_n$ will now converge.

## § 8. Alternating Series.

THEOREM. *The series*

$$u_0-u_1+u_2-\cdots,\qquad 0\leqq u_n,\qquad m\leqq n,$$

*converges if*

i) $\qquad u_{n+1}\leqq u_n,\qquad m\leqq n;$

ii) $\qquad \lim\limits_{n=\infty} u_n=0.$

*Proof.* Form the sums:

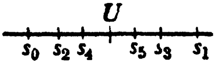
$$s_{2n} = (u_0 - u_1) + \cdots + (u_{2n-2} - u_{2n-1}),$$

$$s_{2n+1} = u_0 - (u_1 - u_2) - \cdots - (u_{2n-1} - u_{2n}).$$

Observe that, since

1) $$s_{2n+1} = s_{2n} + u_{2n},$$

2) $$s_{2n} \leqq s_{2n+1}.$$



Now, $s_{2n}$ steadily increases with $n$, or remains constant; and $s_{2n+1}$ steadily decreases with $n$, or remains constant. It follows, then, that

3) $$s_{2n} \leqq s_1.$$

Consequently, by the Fundamental Principle of § 2, $s_{2n}$ approaches a limit:

$$\lim_{n=\infty} s_{2n} = U_1.$$

Similarly, $s_{2n+1}$ approaches a limit:

$$\lim_{n=\infty} s_{2n+1} = U_2.$$

Finally, because of i) and ii), these two limits are equal:

$$U_1 = U_2.$$

Consequently $s_n$ approaches this common limit, $U$:

$$\lim_{n=\infty} s_n = U,$$

and thus the theorem is proved.

*The Error.* Since $s_n$, with increasing $n$, continually jumps over its limit $U$ (or coincides with it), we see that the error due to breaking the series off at any point, does not exceed numerically the value of the first term in the remainder.

*Examples:*

a) $$1 - \frac{1}{2!} + \frac{1}{3!} - \cdots$$

b) $$1 - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \cdots$$

c) $$\frac{1}{2\log 2} - \frac{1}{3\log 3} + \frac{1}{4\log 4} - \cdots$$

d) $$\frac{1}{\log\log 4} - \frac{1}{\log\log 5} + \cdots$$

How many terms of the first series are needed to obtain the value correct to $\frac{1}{10}$? how many of the last?

### § 9. Series with Positive and Negative Terms at Pleasure.

Let

1) $$u_0 + u_1 + \cdots$$

be any series whatever. Set

$$\sigma_m = v_1 + v_2 + \cdots + v_m,$$
$$\tau_p = w_1 + w_2 + \cdots + w_p,$$

where $v_1$, $v_2$, $\cdots$ denote the positive terms, taken in order as they come, and where the negative terms are $-w_1$, $-w_2$, $\cdots$. Then

$$s_n = u_0 + u_1 + \cdots + u_{n-1}$$

can be written in the form:

2) $$s_n = \sigma_m - \tau_p$$

where, in particular, if no positive terms have appeared, we agree to write $\sigma_0 = 0$; and similarly for $\tau_0$.

As $n$ increases, the series

3) $$v_1 + v_2 + \cdots$$

4) $$w_1 + w_2 + \cdots$$

may converge or they may diverge — it is not necessary to think of the case of a finite number of positive or negative terms as an exception, for any sum may be regarded as an infinite series, all of whose later terms are 0. Thus if 1) is the series

5) $$1 - \frac{1}{2} + \frac{1}{3} - \cdots,$$

both the series 3) and 4) diverge; but 5) converges by § 8. It is clear, however, that if both the series 3) and 4) converge, the variable $s_n$ will approach a limit because of 2), and so the series 1) will converge. We are thus led to the

THEOREM. *A sufficient condition for the convergence of a series*

6) $$u_0 + u_1 + \cdots$$

*is the convergence of the series of absolute values*:

7) $$|u_0| + |u_1| + \cdots$$

*Proof.* Let

$$S_n = |u_0| + |u_1| + \cdots + |u_{n-1}|.$$

Then

$$S_n = \sigma_m + \tau_p.$$

Let $A$ be the value of 7). Then

$$\sigma_m \leqq A, \qquad \tau_p \leqq A,$$

and hence, by § 2, both the series 3) and 4) converge.

A series 1) whose absolute value series converges is said to *converge absolutely*, or be *absolutely convergent*. Other convergent series are called *conditionally convergent*. In the case of these latter series, both the series 3) and 4) diverge. It will be shown later that the terms of an absolutely convergent series may be rearranged at pleasure without changing the value of the series. But if the series converges conditionally, a new series can be constructed from its terms, which will converge toward any preassigned value whatever, or diverge toward $+\infty$, or diverge toward $-\infty$; Chap. VII, § 2.

TEST-RATIO TEST. *If the series*

$$u_0 + u_1 + \cdots, \qquad u_n \neq 0, \qquad m \leqq n,$$

*be given, and if its test-ratio* $u_{n+1} / u_n$ *approaches a limit:*

$$\lim_{n=\infty} \frac{u_{n+1}}{u_n} = r,$$

*then the series*

i) *converges, if* $|r| < 1$;

ii) *diverges, if* $|r| > 1$.

*Proof.* If $|r| < 1$, the series of absolute values converges. For,

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{|u_{n+1}|}{|u_n|},$$

and the left-hand side of this equation approaches a limit, namely, $|r|$.

If, on the other hand, $|r| > 1$, it follows that $u_n$ does not approach 0 as its limit. For, from the proof in § 3, $|u_n|$ does not approach 0.

The more refined test-ratio tests of § 5 help in establishing the absolute convergence of a series, but are useless if the series converges conditionally.

Finally, we observe that it is sufficient for convergence to demand that

$$\left|\frac{u_{n+1}}{u_n}\right| < \gamma < 1, \qquad m \leqq n,$$

where $\gamma$ is a constant; i.e. not to impose the restriction that a limit exists. But the limit form is the one that is useful in practice, and for that reason should be given first place.

COMPARISON TEST. *Let*

$$u_0 + u_1 + \cdots$$

*be a series to be tested for convergence, and let*

$$a_0 + a_1 + \cdots, \qquad a_n \neq 0, \qquad m \leqq n,$$

*be an absolutely convergent series. If* $u_n/a_n$ *approaches a limit:*

$$\lim_{n=\infty} \frac{u_n}{a_n} \text{ exists,}$$

*then the u-series converges absolutely.*

For, since $u_n/a_n$ approaches a limit, it follows that $|u_n|/|a_n|$ must also approach a limit. Denote the latter limit by $L$, and choose $L' > L$. Then, from a definite point on,

$$\frac{|u_n|}{|a_n|} < L', \qquad \mu \leqq n.$$

Hence

$$|u_n| < L'|a_n|.$$

The series whose general term is this last expression converges, and this fact yields the proof.

DIVERGENCE TEST. *Let*

$$u_0 + u_1 + \cdots$$

*be a series to be tested, and let*

$$a_0 + a_1 + \cdots, \qquad 0 < a_n, \qquad m \leqq n,$$

*be a divergent series. If* $u_n/a_n$ *approaches a limit not* 0:

$$\lim_{n=\infty} \frac{u_n}{a_n} = L \neq 0,$$

*then the u-series diverges.*

## EXERCISES

Test the following series for convergence:

1. $$\frac{1\cdot 2}{100^2}+\frac{1\cdot 2\cdot 3}{100^3}+\frac{1\cdot 2\cdot 3\cdot 4}{100^4}+\cdots$$

2. $$\frac{2^{100}}{2}+\frac{3^{100}}{2^2}+\frac{4^{100}}{2^3}+\cdots$$

3. $$\frac{3}{5^3}+\frac{3^2}{10^3}+\frac{3^3}{15^3}+\cdots$$

4. $$\frac{1}{3}+\frac{1\cdot 2}{3\cdot 5}+\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}+\cdots$$

5. $$\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\cdots$$

6. $$\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2+\cdots$$

7. $$\left(\frac{1}{2}\right)^3+\left(\frac{1\cdot 3}{2\cdot 4}\right)^3+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3+\cdots$$

8. $$\frac{\cos x}{1}-\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}-\cdots$$

9. $$1+2x+3x^2+4x^3+\cdots$$

10. $$x-\frac{x^2}{2!}+\frac{x^3}{3!}-\cdots$$

11. $$\frac{x}{2}-\frac{1}{\sqrt{2}}\,\frac{x^2}{2^2}+\frac{1}{\sqrt{3}}\,\frac{x^3}{2^3}-\cdots$$

12. $$x-\frac{x^3}{\sqrt{3}}+\frac{x^5}{\sqrt{5}}-\cdots$$

13. $$\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{a+r}}+\frac{1}{\sqrt{a+2r}}-\cdots,\qquad 0<a.$$

14. $$\log(1+r)+\log(1+r^2)+\log(1+r^3)+\cdots,\qquad 0\leqq r.$$

15. $\log\left(1+\frac{x}{1}\right)+\log\left(1+\frac{x}{2}\right)+\log\left(1+\frac{x}{3}\right)+\cdots, 0 \leqq x.$

16. $\log\left(1-\frac{x^2}{1}\right)+\log\left(1-\frac{x^2}{2^2}\right)+\log\left(1-\frac{x^2}{3^2}\right)+\cdots.$

**§ 10. Infinite Products.** Let $f_0, f_1, \cdots$ be any set of numbers proceeding according to a definite law. Form the product

$$p_n = f_0 \cdot f_1 \cdots f_{n-1}.$$

If $f_n \neq 0$, $m \leqq n$, and the variable

$$p_{m,r} = f_m \cdot f_{m+1} \cdot \cdots \cdot f_{m+r-1}$$

approaches a limit different from 0:

$$\lim_{r=\infty} p_{m,r} = P_m \neq 0.$$

then the *infinite product*

$$f_0 \cdot f_1 \cdots$$

is said to *converge* and the number

$$P = \lim_{n=\infty} p_n$$

is assigned to it as its *value*:

$$P = f_0 f_1 f_2 \cdots.$$

For example, the infinite product

$$\frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdots$$

converges, since the product

$$p_{1,r-1} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdots\left(\frac{r-1\cdot r+1}{r\cdot r}\right) = \frac{1}{2}\,\frac{r+1}{r}$$

approaches a limit different from 0, and so

$$\frac{1}{2} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdots$$

On the other hand, the infinite product

$$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots$$

diverges. For here

$$p_{1,r} = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{r}{r+1} = \frac{1}{r+1}$$

approaches the limit 0.

These high handed definitions are surprising. Their justification lies in the demands of practice. There are no infinite products with an infinite number of vanishing factors in any known domain of mathematics. To admit such for consideration would complicate the theory — *cui bono?* — Again, to admit as convergent an infinite product which vanishes without any one of its factor's vanishing would break with the analogy with the products of ordinary algebra, For these reasons, and others, we have thrown such infinite products into the discard of divergent products.

From the definition of convergence follows immediately that the factors of a convergent infinite product ultimately become and remain positive.

THEOREM. *A necessary and sufficient condition for the convergence of an infinite product:*

$$f_0 \cdot f_1 \cdot f_2 \cdot \cdots$$

*lies in the convergence of the infinite series:*

$$\log f_m + \log f_{m+1} + \cdots,$$

*where m is suitably chosen.*

a) The condition is necessary. Hypothesis: the product converges, or

$$p_{m,r} = f_m \cdot f_{m+1} \cdot \cdots \cdot f_{m+r-1}$$

approaches a limit not 0; moreover, $m$ may and shall be so taken that $f_n > 0$, $m \leqq n$. Conclusion: the series converges. Proof: let

$$s_r = \log f_m + \log f_{m+1} + \cdots + \log f_{m+r-1}.$$

Then

$$s_r = \log p_{m,r}.$$

Since $p_{m,r}$ approaches a limit $P_m > 0$, and since $\log x$ is a continuous function, it follows that $\log p_{m,r}$ approaches a limit, and hence the series converges. We observe in passing that

$$\log P_m = \log f_m + \log f_{m+1} + \cdots$$

b) The condition is sufficient. Hypothesis: the series converges; i.e. lim $s_r$ exists. Conclusion: the product converges. Since

$$p_{m,r} = e^{S_r}$$

and since $e^x$ is a continuous function, it follows that $e^{S_r}$ approaches a limit:

$$\lim_{r=\infty} e^{S_r} = e^{S_m},$$

where

$$S_m = \log f_m + \log f_{m+1} + \cdots .$$

Finally, this limit cannot be 0 because the exponential function is positive for all values of the argument.

COROLLARY. *A necessary condition for the convergence of an infinite product is, that*

$$\lim_{n=\infty} f_n = 1.$$

For this reason it is often convenient to set

$$f_n = 1 + a_n.$$

Thus the infinite product appears in the form:

$$(1 + a_0)(1 + a_1)\cdots$$

CONVERGENCE TEST. *A sufficient condition for the convergence of the infinite product*

$$(1 + a_0)(1 + a_1)\cdots$$

*is the absolute convergence of the series:*

$$a_0 + a_1 + \cdots .$$

We may omit any factors in which $a_n = 0$, since such influence the convergence of neither the product nor the series.

Consider the series:

$$\log(1 + a_0) + \log(1 + a_1) + \cdots$$

Apply the Comparison Test of § 9. Since

$$\lim_{x=0} \frac{\log(1 + x)}{x}$$

exists, it follows that $\log(1 + a_n)/a_n$ approaches a limit as $n$ becomes infinite, and so the series of logarithms converges. Hence the infinite product converges.

It is not true that the mere convergence of the $a$-series will insure the convergence of the product. For example, the product

$$\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{4}}\right)\cdots$$

is easily seen to diverge by use of the series of logarithms. But the series

$$-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\cdots$$

converges. Nevertheless, if the $a$'s be restricted not to change sign we can obtain a divergence test.

DIVERGENCE TEST. *If*

$$a_0 + a_1 + \cdots$$

*is a divergent series whose terms from a definite point on are all of one sign, the infinite product*

$$(1 + a_0)(1 + a_1)\cdots$$

*diverges.*

The notation for an infinite product is:

$$\prod_{n=0}^{\infty} f_n.$$

EXERCISES

Test the following infinite products for convergence.

1. $$\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{4^2}\right)\cdots$$

2. $$\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\cdots$$

3. $$\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right).$$

4. $$\prod_{n=1}^{\infty}\left(1-\frac{4r^2}{(2n+1)^2}\right).$$

5. $$\prod_{n=1}^{\infty}\left(1-\frac{x}{n}\right)e^{\frac{x}{n}}.$$

6. $$\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)\left(1+\frac{1}{n}\right)^{-x}$$

7. Study the Binomial Series:

$$1 + mx + \frac{m(m-1)}{1\cdot 2}x^2 + \cdots,$$

determining its convergence in all cases.

8. Show that the infinite product

$$\prod_{n=1}^{\infty}\left(1 - \frac{(1+r)n + (c-ab)}{(a+n)(b+n)}\right), \quad a, b \neq -1, -2, -3, \cdots,$$

diverges toward $+\infty$ when $r < -1$; converges, when $r = -1$; and diverges toward 0 when $-1 < r$.

## § 11. The Hypergeometric Series:

$$F(a, b, c, x) = 1 + \frac{a\cdot b}{c\cdot 1}x + \frac{a(a+1)\,b(b+1)}{c(c+1)\,1\cdot 2}x^2 + \cdots$$

This series includes some of the important elementary functions; but it also defines a class of more general functions, which arise not infrequently in practice and whose properties have been studied by means of the theory of functions of a complex variable.

Some simple specializations of the series are the following,

$$(1+x)^n = F(-n, b, b, -x);$$

$$\log(1+x) = x\,F(1, 1, 2, -x);$$

$$\tan^{-1} x = x\,F\left(\frac{1}{2}, 1, \frac{3}{2}, -x^2\right);$$

$$e^x = \lim_{b=\infty} F\left(1, b, 2, \frac{x}{b}\right).$$

We propose to discuss the convergence of the series in all cases.

First of all, we must impose the restriction: $c \neq 0, -1, -2, \cdots$, for otherwise no series is defined.

Next, when either $a$ or $b$ is $0, -1, -2, \cdots$, the series breaks off with a finite number of terms, thus reducing to a polynomial, and so, considered as an infinite series, converges for all values of $x$.

There remain, then, the cases:

$$a, b, c \neq 0, -1, -2, \cdots$$

For what values of $x \neq 0$ will the series converge?

The general term is easily formulated:

$$\frac{a(a+1)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)\,1\cdot 2\cdots n}\,x^n.$$

Thus

$$\frac{u_{n+1}}{u_n}=\frac{(a+n)(b+n)}{(c+n)(1+n)}\,x.$$

Its limit is $x$, and so the series converges absolutely in the interval

$$-1<x<1,$$

and diverges outside this interval.

There are still two points in doubt, namely $x=+1,\ -1$, for each of which the test-ratio $|u_{n+1}|\,/\,|u_n|$ approaches the limit 1.

*The Point* $x=+1$. Here, we can use the Second Test-Ratio Test in Kummer's form, § 6. Since

$$\frac{u_n}{u_{n+1}}=\frac{(c+n)(1+n)}{(a+n)(b+n)},$$

we have:

$$n\,\frac{u_n}{u_{n+1}}-(n+1)=\frac{(n+1)\left[(c-a-b)n-ab\right]}{(a+n)(b+n)};$$

and thus $\sigma$ is seen to have the value:

$$\sigma=c-a-b.$$

Hence if

$$0<c-a-b,\qquad\text{the series converges;}$$

$$c-a-b<0,\qquad\text{the series diverges.}$$

When $c-a-b=0$, the test fails, and we proceed to the next test. Here,

$$n\log n\,\frac{u_n}{u_{n+1}}-\overline{n+1}\,\log\overline{n+1}=$$

$$(n+1)\left[\frac{n(c+n)}{(a+n)(b+n)}\log n-\log\overline{n+1}\right]=$$

$$-(n+1)\left[\log(n+1)-\log n+\frac{ab\log n}{(a+n)(b+n)}\right]$$

The limit of this variable is seen to be $-1$, and consequently the series diverges.

To sum up, then: In the case $x = +1$ we have

i) Convergence, when $0 < c - a - b$;

ii) Divergence, when $c - a - b \leqq 0$.

*The Point* $x = -1$. Here

$$\frac{u_{n+1}}{u_n} = -\frac{(a+n)(b+n)}{(c+n)(1+n)},$$

and the series ultimately becomes an alternating series. When

$$0 < c - a - b,$$

the series converges absolutely. When

$$c - a - b \leqq 0,$$

we have:

$$\frac{|u_{n+1}|}{|u_n|} = 1 - \frac{(1+r)n + (c - ab)}{(c+n)(1+n)},$$

where

$$r = c - a - b \leqq 0.$$

Let

$$a_n = -\frac{(1+r)n + (c - ab)}{(c+n)(1+n)}.$$

Then

$$\frac{|u_{m+k}|}{|u_n|} = (1 + a_m)(1 + a_{m+1})\cdots(1 + a_{m+k-1}).$$

The infinite product

$$\prod_{n=m}^{\infty}(1 + a_n)$$

diverges toward $+\infty$ when $r < -1$, and consequently $u_n$ does not approach 0 as its limit; the $u$-series diverges.

When $r = -1$, the infinite product converges, and again the $u$-series diverges, for the same reason as before.

But when $r > -1$, the infinite product diverges toward 0. We have:

i) $$\frac{|u_{n+1}|}{|u_n|} < 1, \qquad \mu \leqq n;$$

ii) $$\lim_{n=\infty} u_n = 0.$$

Consequently the $u$-series converges conditionally.

To sum up, then:— When $x = -1$, we have

i) Absolute Convergence, when $0 < c - a - b$;
ii) Conditional Convergence, when $-1 < c - a - b \leqq 0$;
iii) Divergence, when $c - a - b \leqq -1$.

The student should supplement his work on this chapter with a fresh and careful study of the chapter on Infinite Series in the Author's *Introduction to the Calculus* (Chap. XIV) and also with a renewed study of the chapter on *Indeterminate Forms* in the *Advanced Calculus* (Chap. X.)

# Chapter II

## The Number System

**§1. The Problem.** We take the natural numbers 1, 2, 3, ⋯, for granted. We think of them as collections of individuals, like a bag of marbles; but we define them as *marks*, like 1, 2, $a$, $b$, etc. The representation of such a number by means of a base, as:

$$347 = 3 \times 10^2 + 4 \times 10 + 7,$$

is not a part of the concept, number.

We assume the idea of addition as known, whereby out of the two numbers, $a$ and $b$, a third is formed:

$$c = a + b.$$

We think of two bags of marbles emptied into a third bag.

Furthermore, we define subtraction as solving the equation:

$$a + x = b, \qquad \text{when} \qquad a < b,$$

and write:

$$x = b - a.$$

Multiplication we also take for granted. By $ab$ we mean the number $b$ added $a$ times:

$$ab = b + b + \cdots + b \ (a \text{ times})$$

Divison consists in solving the equation

$$ax = b,$$

when $b$ is a multiple of $a$.

It will also be convenient to point out that both subtraction and division, when possible, are unique; and furthermore to adjoin the theorem (proved by Euclid's algorithm of the greatest common divisor) that a natural number can be represented in one and (except for the order) but one way as the product of its prime factors:

$$a = p_1^{l_1} \, p_2^{l_2} \, \ldots \, p_\nu^{l_\nu} \, .$$

A corallary of this theorem should also be noted:— If $a$ and $b$ are relatively prime to each other, and if $ac$ is divisible by $b$, then $c$ is divisible by $b$.

*Inequality.* We also assume as known what is meant by $a < b$, $a > b$, and point out that

i) if $a < b$, then $b > a$, and conversely;
ii) if $a < b$, $b < c$, then $a < c$;
iii) if* $a < b$ and $c < d$, then $a + c < b + d$;
iv) if** $a < b$ and $c < d$, then $ac < bd$;
v) if $a < b$, then $na > b$, for a suitable $n$.

*The Formal Laws of Algebra.* For the numbers and the processes above considered the following laws hold without exception:

I. $A + B = B + A$, Commutative Law for Addition;
II. $A + (B + C) = (A + B) + C$, Associative Law for Addition;
III. $A B = B A$, Commutative Law for Multiplication;
IV. $A(B C) = (A B) C$, Associative Law for Multiplication;
V. $A(B + C) = A B + A C$, Distributive Law.

By an *algebraic equation* is meant an equation of the form:†

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = b,$$

where the coefficients are any numbers of the particular domain in question.

This equation includes the equation

$$1) \qquad a + x = b,$$

which we met in defining subtraction; and also

$$2) \qquad ax = b,$$

appearing in division.

We now proceed to extend the number system so that Equation 2) will always admit a solution.

*The Concept: Number.* What is a "number"? We have been so used to the numbers 1, 2, 3, 4, $\cdots$ from earliest childhood, that we take them for granted; and also the simpler fractions, like $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$.

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* If $a \leqq b$ and $c \leqq d$, where at least one of the upper signs holds, then

$$a + c < b + d.$$

** If $a \leqq b$ and $c \leqq d$, where at least one of the upper signs holds, then

$$ac < bd.$$

† Later, when the number system has been enlarged to include the number 0, we shall set $b = 0$ and require that $a_0 \neq 0$.

But what is $\sqrt{2}$, $\sqrt{-1}$? The answer lies in the concept of a class of objects, or as it is sometimes technically called: a *logical class*. The whole numbers are a case in point; for, any two numbers, $a$ and $b$, of this class are either equal or unequal. And for the objects of this class we have two processes — addition and multiplication, whereby respectively any two objects, $a$ and $b$, determine a third $c\,(= a + b$ or $= ab)$.

Now, we proceed to enlarge this class — or, better, to define a more comprehensive class, containing the whole numbers as a subclass — our object being to provide ourselves with a number system, i.e. a logical class, in which addition and multiplication are defined, such that, whatever numbers $a$ and $b$ may be in the new system, division is always possible; i.e. Equation 2) always has a solution.

**§ 2. Fractions.** Consider the logical class, or set of elements,

$$(m, n),$$

where $m$, $n$ are any natural numbers. The element is the *mark* $(m, n)$ itself, and we will speak of it as a *number* and refer to it by a single letter, as

$$A = (a_1, a_2), \qquad B = (b_1, b_2), \qquad X = (x, y), \quad \text{etc.}$$

*Equality.* Two numbers, $A = (a_1, a_2)$, $B = (b_1, b_2)$ shall be said to be *equal*, when

$$a_1 b_2 = a_2 b_1.$$

We write

$$A = B.$$

It follows from the definition that if

$$A = B$$

then

$$B = A;$$

and if

$$A = B, \qquad B = C,$$

then

$$A = C.$$

If $A = (m, n)$, then does $A$ also $= (\lambda m, \lambda n)$, where $\lambda$ is any natural number; and also, in case $m$ and $n$ are both divisible by $\lambda$:

$$A = \left(\frac{m}{\lambda}, \frac{n}{\lambda}\right).$$

Let $\lambda$ be so chosen that

$$m' = \frac{m}{\lambda}, \qquad n' = \frac{n}{\lambda}$$

are prime to each other. Then

$$A = (m', n')$$

is said to be in *reduced*, or *normal form*.

If $B = (b_1, b_2)$ is any number which is equal to $A$,

$$A = B,$$

then

$$b_1 = \mu\, m', \qquad b_2 = \mu\, n',$$

where $\mu$ is a natural number.

*Inequality.* We say that

$$(m, n) < (m', n'),$$

when

$$mn' < m'n.$$

Relations i)—v), § 1, are then seen to hold in the present system.

*Addition.*[*] By the *first combination* (*erste Verknüpfung*) we mean the following. If $A = (a_1, a_2)$, $B = (b_1, b_2)$, then

$$A \oplus B = (a_1 b_2 + a_2 b_1,\ a_2 b_2).$$

(Read: "$A$ with $B$".) A "combination" is essentially a function of two variables: $f(A, B)$, having properties which result from the specific definition.

The Commutative and the Associative Law follow at once:

I. $$A \oplus B = B \oplus A;$$

II. $$A \oplus \{ B \oplus C \} = \{ A \oplus B \} \oplus C.$$

*Multiplication.* By the *second combination* we mean:

$$A \otimes B = (a_1 b_1,\ a_2 b_2).$$

It is also commutative and associative:

III. $$A \otimes B = B \otimes A,$$

IV. $$A \otimes \{ B \otimes C \} = \{ A \otimes B \} \otimes C.$$

Finally, the Distributive Law holds:

V. $$A \otimes \{ B \oplus C \} = \{ A \otimes B \} \oplus \{ A \otimes C \}.$$

Proof, by direct computation.

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[*] It would be better to consider first multiplication, as being simpler. But such an order is less in harmony with the procedure in the later cases.

*Division.* We are now in a position to prove that division is always possible and unique. Given the equation:

1) $$A \otimes X = B,$$

to solve for $X = (x_1, x_2)$.

Here, we have as a necessary and sufficient condition:

2) $$a_1\, b_2\, x_1 = a_2\, b_1\, x_2.$$

Obviously, one solution of 2) is:

$$x_1 = a_2\, b_1, \qquad x_2 = a_1\, b_2$$

Let

$$x_1 = \lambda x_1', \qquad x_2 = \lambda x_2',$$

where $x_1'$, $x_2'$ are prime to each other. And now the number $X = (x_1', x_2')$ is seen to be a solution of 1), and the only one, any two numbers that are equal being regarded as identical. For the most general solution of 2) is $x_1 = \mu\, x_1'$, $x_2 = \mu\, x_2'$, and 2) is both necessary and sufficient.

*Idemfactor.* There exists a number $I = (1, 1)$ such that

$$I \otimes A = A, \qquad A \otimes I = A,$$

where $A$ is any number of the system; and only one such number.

*The Natural Numbers as a Sub-Class.* Within the present class of numbers there is a sub-class which can be identified with the natural numbers. If we associate the number $(a, 1)$ with $a$:

$$(a, 1) \backsim a,$$

then the first and second combinations will also correspond respectively to addition and multiplication. For, if

$$(a, 1) \oplus (b, 1) = (c, 1)$$

then
$$a + b = c;$$

and if
$$(a. 1) \otimes (b, 1) = (c, 1),$$

then
$$ab = c.$$

We can, therefore, replace the number $(a, 1)$ by $a$ in all computations, and furthermore write, as a matter of notation:

$$(m, n) = \frac{m}{n}\cdot$$

Thus the present class reduces to the natural numbers and the positive fractions.

The "first combination" and the "second combination", together with the notation $\oplus$ and $\otimes$, have now served their purpose of setting forth precisely in what "addition" and "multiplication" consist as applied to fractions. We may, therefore, henceforth discard them, and write $+$ and $\times$ as signs for the processes we have so carefully defined; e.g.

$$\frac{5}{12}+\frac{16}{35}=\frac{175+192}{420}=\frac{367}{420},$$

$$\frac{5}{12}\times\frac{16}{35}=\frac{80}{420}=\frac{4}{21}.$$

Between any two numbers, $a$ and $b$, $(a < b)$ of the present system there lies a number of the system. For, the number

$$c=\frac{a+b}{2}$$

has the property that

$$a < c < b.$$

Hence there is an infinite set of numbers lying between $a$ and $b$.

There is no least number. For, if $a$ be any number, then $\frac{1}{2}a$ also is a number, and $\frac{1}{2}a < a$.

By the *distance* between $a$ and $b$, where $a < b$, shall be meant the number $b - a$.

If $a$ be any number of the system, then there are numbers larger than $a$ and arbitrarily near $a$; and also numbers less than $a$ and arbitrarily near $a$. For, let $\epsilon$ be chosen arbitrarily small and, in particular, let $\epsilon < a$. Then the number $a - \epsilon$ is less than $a$ and distant only $\epsilon$ from $a$. And, similarly, $a + \epsilon > a$ and distant only $\epsilon$ from $a$.

Finally, if $a$ be a number of the system, however small; and $G$, a number, however large, then a natural number $m$ can be found such that

$$ma > G.$$

For, let

$$a=\frac{p}{q}, \qquad G=\frac{P}{Q}$$

(both taken, for definiteness, in reduced form). Then the inequality we wish to establish is, that

$$\frac{mp}{q} > \frac{P}{Q}.$$

By definition, this inequality holds if

$$mpQ > qP.$$

Here $pQ$ and $qP$ are natural numbers, and an $m$ can always be found to satisfy this condition — e.g. $m = qP + 1$.

§ 3. **Negative Numbers.** Consider a new logical class, or class of elements, $(a, b)$, where $a, b$ are fractions.* Two of these numbers, $(a, b)$ and $(a', b')$, shall be said to be *equal*:

$$(a, b) = (a', b'),$$

if

$$a + b' = a' + b.$$

It follows that

$$(a + \lambda,\ b + \lambda) = (a, b),$$

where $\lambda$ is any fraction; and also

$$(a - \lambda,\ b - \lambda) = (a, b),$$

so far as both $a - \lambda$ and $b - \lambda$ have a meaning. Moreover, these are obviously the only numbers equal to $(a, b)$.

An arbitrary number of the system can evidently be reduced to one or the other of the forms $(a, 1)$, $(1, a)$.

*Inequality.* By definition:

$$(a, b) < (a', b')$$

if

$$a + b' < a' + b.$$

*Addition.* The first combination shall be defined as follows. If $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then

$$A \oplus B = (a_1 + b_1,\ a_2 + b_2).$$

---

* The component elements, $a$, $b$, etc. might equally well be natural numbers; i.e. we can introduce the negative whole numbers in this way before proceeding to fractions. On the other hand, we might postpone the introduction of the negative numbers till the positive irrationals had been defined, and then $a$, $b$, etc. would be positive irrationals. — The notation ( ) has, of course, nothing to do with the ( ) of § 2, which has now been discarded. Let the student replace it by [ ], writing $[a, b]$, etc., if this form is more agreeable to him.

The Commutative Law,

$$A \oplus B = B \oplus A,$$

and the Associative Law:

$$A \oplus \{B \oplus C\} = \{A \oplus B\} \oplus C,$$

are seen to hold.

*Subtraction.* Let it be required to solve the equation:

$$A \oplus X = B.$$

Let $X = (x_1, x_2)$. Then

$$(a_1 + x_1,\ a_2 + x_2) = (b_1, b_2).$$

Hence

$$a_1 + x_1 + b_2 = a_2 + x_2 + b_1,$$

and so

$$x_1 = a_2 + b_1$$
$$x_2 = a_1 + b_2$$

These equations are both necessary and sufficient. Hence subtraction — defined as the inverse of addition — is always possible and unique. The difference, $X$, is denoted as follows:

$$X = B \ominus A.$$

*The Number* 0. The number

$$A_0 = (a, a)$$

has the property that

$$A_0 \oplus A = A, \qquad A \oplus A_0 = A,$$

where $A$ is any number of the system. It is denoted by 0, or $(a, a) = 0$.

The number $A'$, called the *negative* of $A$, is defined by the equation:

$$A \oplus A' = 0.$$

If $A = (a, b)$, then $A' = (b, a)$.

Furthermore,

$$A' = 0 \ominus A.$$

It thus becomes natural to attach to the sign $\ominus$ a new meaning, *defining* $\ominus A$ as the negative of $A$. Observe that the sign $\ominus$ is now used in two senses — once, as a functional symbol in a function of two variables,

$$f(A,B) = B \ominus A;$$

and again as a functional symbol in a function of one variable,

$$\varphi(A) = A' = \ominus A.$$

This explains the meaning of $-a$ in ordinary algebra.* It is the negative, $a'$, of $a$; the number which, when added to a will give 0. But $a$ may be positive, negative, or 0, and $-a$ is not necessarily a negative number.

The equation

$$A \oplus X = B$$

can be solved conveniently by means of $A'$, the negative of $A$. For, a necessary condition for $X$ is, that

$$A' \oplus A \oplus X = A' \oplus B.$$

Hence

$$0 \oplus X = A' \oplus B,$$

or

$$X = B \oplus A'.$$

Conversely, this condition is sufficient.

*The Law of Signs.* To avoid the ambiguity in the use of the sign $\ominus$, replace $\ominus$ in the second sense by $\ominus'$. Thus if $A'$ is the negative of $A$, we will write:

$$A' = \ominus' A.$$

Observe that

$$\ominus'(\ominus' A) = A; \qquad A \oplus (\ominus' B) = A \ominus B; \qquad A \ominus (\ominus' B) = A \oplus B.$$

We find by direct computation that

$$\ominus'(A \oplus B) = \ominus' A \oplus (\ominus' B) = \ominus' A \ominus B,$$

$$\ominus'(A \ominus B) = \ominus' A \oplus B;$$

$$\ominus'(\ominus' A \oplus B) = A \ominus B;$$

$$\ominus'(\ominus' A \ominus B) = A \oplus B.$$

These identities justify the ordinary rules for the use of the minus sign in Algebra. They show clearly the two meanings of that sign in such transformations as

$$-(a - b + c) = -a + b - c.$$

Here, the first term on the right, $-a$, is the negative, $a'$, of $a$. The last minus sign is capable of two interpretations:

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* In the lines that follow, our references to ordinary algebra are purely for illustrative purposes. The systematic connection with ordinary algebra will be made at the close of the paragraph.

i) it may be the $\ominus$ of

$$(-a+b)\ominus c;$$

ii) it may be the $\ominus'$ of the negative of $c$, $c' = \ominus' c$; the plus sign being omitted:

$$(-a+b)\oplus(\ominus' c) = -a+b\ominus' c.$$

*Multiplication.* The second combination is defined as follows. If $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then

$$A\otimes B = (a_1 b_1 + a_2 b_2,\ a_1 b_2 + a_2 b_1).$$

It is at once obvious that the Commutative Law holds:

$$A\otimes B = B\otimes A.$$

The Associative Law,

$$A\otimes\{B\otimes C\} = \{A\otimes B\}\otimes C.$$

is also true. For, compute the left-hand side; it is:

$$(a_1 b_1 c_1 + a_1 b_2 c_2 + a_2 b_1 c_2 + a_2 b_2 c_1,$$
$$a_1 b_1 c_2 + a_1 b_2 c_1 + a_2 b_1 c_1 + a_2 b_2 c_2).$$

Next, observe that, on the right,

$$\{A\otimes B\}\otimes C = C\otimes\{B\otimes A\}.$$

It is, therefore, enough to note that the above expression is unchanged when the $a$'s and the $c$'s are interchanged.

Finally, the Distributive Law,

$$A\otimes\{B\oplus C\} = \{A\otimes B\}\oplus\{A\otimes C\},$$

is shown to hold by direct computation.

*The Idemfactor.* The number

$$I = (a+1, a)$$

has the property that

$$A\otimes I = A, \qquad I\otimes A = A.$$

It will be shown presently that it is the only number that has this property for all $A$'s, and that even for a particular $A \neq 0$ it is unique.

This number is, for multiplication, what 0 is for addition:

$$A\oplus 0 = 0\oplus A = A.$$

*Divison.* Let it be required to solve the equation

$$A\otimes X = B.$$

If $A = (a_1, a_2)$ be any number $\neq 0$; i.e. $a_1 \neq a_2$; then there exists a number $A'$ such that

$$A \otimes A' = I, \quad \text{or} \quad A' \otimes A = I.$$

We wish, namely, to solve the equation:

$$(a_1, a_2) \otimes (x_1, x_2) = (\lambda + 1, \lambda),$$

where $\lambda$ is arbitrary. Thus

$$\begin{aligned} a_1 x_1 + a_2 x_2 &= \lambda + 1, \\ a_2 x_1 + a_1 x_2 &= \lambda. \end{aligned}$$

Hence either i):

$$\begin{cases} (a_1^2 - a_2^2)\, x_1 = \lambda\,(a_1 - a_2) + a_1 \\ (a_1^2 - a_2^2)\, x_2 = \lambda\,(a_1 - a_2) - a_2 \end{cases}$$

if $a_1 > a_2$ and $\lambda$ is so chosen that

$$\lambda\,(a_1 - a_2) > a_2.$$

Or ii): if $a_1 < a_2$, then

$$\begin{cases} (a_2^2 - a_1^2)\, x_1 = \lambda\,(a_2 - a_1) - a_1 \\ (a_2^2 - a_1^2)\, x_2 = \lambda\,(a_2 - a_1) + a_2 \end{cases}$$

$\lambda$ being so chosen that

$$\lambda\,(a_2 - a_1) > a_1.$$

When, however, $a_1 = a_2 = c$, the equations reduce to

$$\begin{cases} c\,x_1 + c\,x_2 = \lambda + 1 \\ c\,x_1 + c\,x_2 = \lambda. \end{cases}$$

These equations never admit a solution.

Returning, now, to the equation

$$A \otimes X = B,$$

assume that $A \neq 0$. Then a necessary condition for $X$ is, that

$$A' \otimes A \otimes X = A' \otimes B$$

or

$$I \otimes X = A' \otimes B$$

or

$$X = B \otimes A'.$$

Conversely, this condition is sufficient.

Thus division is always possible when $A \neq 0$. Moreover, it is unique. In particular, then, $I$ is unique.

Finally, if the product of two numbers is 0, then one of the factors is 0. For, if $B = 0$, $A \neq 0$, then $X = 0$.

*Relation to the Fractions.* The present number system contains a sub-class of numbers, namely, the numbers $(a + 1, 1)$, which, together with the four species, are holoedrically isomorphic with the fractions. By this we mean that if we relate

$$A = (a + 1, 1) \quad \text{with} \quad a,$$

and so

$$B = (b + 1, 1) \quad \text{with} \quad b;$$

then

$$A \oplus B = (a + b + 2, 2)$$

will correspond to $a + b$ and

$$A \otimes B = (ab + 1, 1)$$

will correspond to $ab$. Moreover, if

$$A < B, \qquad \text{then} \qquad a < b.$$

Thus the numbers $(a + 1, 1)$ of this sub-class obey precisely the same laws of combination (equality, inequality, the four species) as do the fractions, § 2, and we may, then, write interchangeably $(a + 1, 1)$ and $a$, $\oplus$ and $+$, $\otimes$ and $\times$, etc.

The remaining numbers of the class, except 0, are the so-called "negatives" (see above) of these. For, any such number can be written in the form:

$$A = (1, a + 1),$$

and its negative, $A'$, is:

$$A' = \ominus' A = (a + 1, 1) = a,$$

or

$$A = \ominus' a = - a.$$

Observe that we are herewith defining $-a$. The number 0 is its own negative: $-0 = 0$.

The numbers $(a, b)$ for which $a > b$ are called *positive numbers*; those for which $a < b$, *negative numbers*. The number 0 is neither positive nor negative.

The numbers of our present system reduce, then, to the fractions, $a = p/q$, of § 2, which we shall henceforth refer to as the *positive fractions*; their negatives, $-a = -p/q$, which we shall call the *negative fractions*; and 0. This number system is called the system of the *commensurable* or *rational numbers*.

The *absolute value* of a positive number or zero is defined as the number itself:

$$|A| = A \quad \text{if } A = (a, b) \text{ and } a \geqq b.$$

The absolute value of a negative number is its negative:

$$|A| = A' = (b, a), \quad \text{if } A = (a, b) \text{ and } a < b.$$

*Inequalities.* The relations of inequality, § 1, i), ii), iii), hold in the present number system; but iv) and v) lapse. Instead of vi) we have, however:

$$\text{iv}') \qquad \begin{cases} \text{if } a < b \text{ and } c > 0, \text{ then } ac < bc; \\ \text{if } a < b \text{ and } c < 0, \text{ then } ac > bc. \end{cases}$$

**§ 4. Irrational Numbers.** *The Cut.* Among the numbers thus far considered — the positive and negative fractions — there is no number that can solve the equation:

$$x^2 = 2.$$

For suppose $x = m/n$ were a root. Then

$$\frac{m^2}{n^2} = 2,$$

or

$$m^2 = 2n^2.$$

Here is a contradiction. For, each prime factor of the left-hand side appears an even number of times, but on the right, 2 appears an odd number of times.

If we consider an arbitrary positive fraction, its square is either less than 2 or greater than 2. In the first case, we put it into the class of numbers whose elements we denote by $a_1$; in the second case we put it into the class of numbers whose elements we denote by $a_2$. Thus we have a partition of all the positive fractions into two classes, and each $a_1$ is less than any $a_2$:

$$a_1 < a_2.$$

Such a partition Dedekind called a *cut*. More generally, let any criterion be given whereby the commensurable numbers fall into two classes, an arbitrary number $a_1$ of the first class being less than any number $a_2$ of the second class. For example, every fraction, $a$, has the property that any other fraction is either less than $a$, and so belongs in the class of the $a_1$, or is greater than $a$, and so belongs in the class of the $a_2$. The number $a$ itself may be assigned to either class. But the class of cuts is more comprehensive than the class of the commensurable numbers. There will be in general no largest $a_1$ nor smallest $a_2$.

Dedekind considered the logical class consisting of all cuts. This is the class of the *irrational numbers*. Let the *mark*

$$(a_1, a_2)$$

refer to such a cut. It — the *mark* — shall be called a *number*, and it may also be denoted by a single letter, as

$$A = (a_1, a_2).$$

Here, the mark $A$ is the number.

*Equality and Inequality*. Two numbers, $A = (a_1, a_2)$ and $B = (b_1, b_2)$, are defined as *equal*:

$$A = B,$$

if $$a_1 \leqq b_2, \qquad b_1 \leqq a_2,$$

no matter how $a_1$, $a_2$, $b_1$, $b_2$ be chosen from their respective classes. At most one of the lower signs can hold.

We say,

$$A < B \quad \text{or} \quad B > A$$

if it be possible to find a particular $b_1 = b_1'$ and a particular $a_2 = a_2'$ such that

$$a_2' < b_1'.$$

*Rational Numbers*. Those numbers $(a_1, a_2)$ for which there is a largest $a_1$ (call it $a$) or a smallest $a_2$ (call it $a$) shall be identified with the rational numbers: $(a_1, a_2) = a$.

Those numbers $(a_1, a_2)$ for which some $a_1$ is positive are called *positive numbers*; those for which an $a_2$ is negative, *negative numbers*. The number $(-a_2, -a_1)$ is the negative of $(a_1, a_2)$, and is written $-(a_1, a_2)$. The number 0 is neither positive nor negative.

By the *absolute value* of $(a_1, a_2)$ is meant the number itself, when it is positive or 0; its negative, when $(a_1, a_2) < 0$. It is denoted by the symbol:

$$|A| = |(a_1, a_2)|.$$

**§5. The Theorem of Continuity.** The present number system has the property which Dedekind characterized as *continuity*. It is embodied in the following

THEOREM. *Let $s_1, s_2, \cdots$ be a set of numbers such that*

$$\text{i)} \qquad s_n \leqq s_{n+1};$$

$$\text{ii)} \qquad s_n \leqq A.$$

*Then there exists a number $U$ such that*

$$\text{a)} \qquad s_n \leqq U, \qquad n = 1, 2, \cdots;$$

$$\text{b)} \qquad \gamma < s_n, \qquad m \leqq n,$$

*where $\gamma$ is any number less than $U$:*

$$\gamma < U,$$

*and $m$ is a corresponding integer.*

*Proof.* Let $a_1$ be a fraction exceeded by some $s_n$ (and so by all later ones). Consider the totality of such fractions. Let $a_2$ be any one of the remaining fractions. Then the number

$$U = (a_1, a_2)$$

has the properties a) and b). For, first, if $U$ did not satisfy Condition a), there would be a number

$$s_{n'} > U,$$

and between $s_{n'}$ and $U$, a fraction $c$:

$$U < c < s_{n'}.$$

Because $c < s_{n'}$, $c$ is an $a_1$. This is impossible because every $a_1 \leqq U$.

Secondly, if Condition *b*) is not fulfilled, then

$$s_n \leqq \gamma$$

for all values of $n$. Choose a fraction, $c$, between $\gamma$ and $U$:

$$\gamma < c < U.$$

Then $s_n < c$, and hence $c$ is an $a_2$. This is impossible because every $a_2 \geqq U$. The proof is now complete.

*Decimal Fractions.* Let $M$ be any integer — positive, negative, or 0 — and let $a_1, a_2, \cdots$ $(0 \leqq a_n \leqq 9)$ be any set of whole numbers proceeding according to a definite law. Then the *mark*

$$M.\, a_1 a_2 \cdots,$$

which is understood as an abbreviation for the *mark*:

$$M + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots,$$

is know as a *decimal fraction*. Thus $3.1415926 \cdots$ is a decimal fraction. If we set

$$s_n = M + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n}$$

and identify this number with the $s_n$ of Dedekind's Theorem, the *value* of the decimal is defined as the corresponding number, $U$.

Conversely, to an arbitrary number $A$ corresponds a decimal fraction whose value is $A$. For, let $M$ be the largest integer which is less than $A$. Choose $a_1$ so that

$$M + \frac{a_1}{10} < A, \qquad A \leqq M + \frac{a_1 + 1}{10},$$

and so on.

## § 6. Convergence. Limits. The Fundamental Theorem.*

By the *neighborhood* of a point $A = (a_1, a_2)$ are meant the numbers $x$,

$$\gamma_1 < x < \gamma_2,$$

where $\gamma_1 < A$ and $\gamma_2 > A$ are arbitrary. By the *ε-neighborhood* of $A$ is meant the neighborhood, where $\epsilon$ is a positive fraction and

$$\gamma_1 = (a_1 - \epsilon, a_2 - \epsilon), \qquad \gamma_2 = (a_1 + \epsilon, a_2 + \epsilon).$$

Let $M_1, M_2, \cdots$ be a succession of point sets** having the following property. To a positive fraction $\epsilon$ chosen at pleasure there corresponds an integer $m$ and a point $\xi_m$ of $M_m$ such that the

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* It may seem an error of judgment, in these Lectures, to introduce at this point a theorem which, in the scientific development of the student, clearly belongs in the chapter on Point Sets. Still, scientific perspective may on occasion become more important than the principles of pedagogy, and the fact that this theorem is more fundamental than even the Four Species for irrational numbers entitles it to rank these processes. The beginner will do well to read this theorem and its proof with an open mind, but not, at this stage, to give it more thought than his imagination readily supplies. He can come back to it after finishing the chapter on Point Sets, but he should think of its place in the science as here, rather than later.

** cf. Chap. III, § 1. It is merely the conception or definition of a point set with which we are here concerned.

$\epsilon$-neighborhood of $\xi_m$ contains the set $M_n$, if $n \geqq m$. The point sets $M_n$ are then said to *converge* as $n = \infty$.

THEOREM 1. *Let $M_1, M_2, \cdots$ be a convergent sequence of point sets. Then there exists a number $U$ such that an arbitrary $\epsilon$-neighborhood of $U$ contains all the later $M_n$ $(n \geqq m)$.*

*Proof.* Let $a_1$ be a fraction that is exceeded by some point of $M_n$ for all values of $n$ from a definite point on; i.e. for $n \geqq m$, where $m$ depends on $a_1$. Consider the totality of such fractions $a_1$. Let $a_2$ be any fraction not an $a_1$. Then the $a_1$, $a_2$ define a cut:

$$U = (a_1, a_2).$$

This number $U$ has the properties of the $U$ of the theorem. For, let $\epsilon'$ be a positive fraction $< \frac{1}{2}\epsilon$, and let $\mu$, $\xi'_\mu$ be the numbers $m$, $\xi_m$ corresponding to $\epsilon'$ by the hypothesis of convergence. Then $M_n$ lies in the $\epsilon'$-neighborhood of $\xi'_\mu$ when $n \geqq \mu$. But the points of this neighborhood lie in the $\epsilon$-neighborhood of $U$. For if it extended above the latter, there would be fractions $a_1 > U$; and if it extended below the latter, there would be fractions $a_2 < U$.

*Definition of a Limit.* The number $U$ is defined as the *limit* of the numbers $x$ constituting $M_n$, as $n$ becomes infinite.

THEOREM 2. *Let $M_1, M_2, \cdots$ be a convergent sequence of point sets, and let $N_1, N_2, \cdots$ be a second sequence, the points of $N_n$ being all contained in $M_n$. Then the second sequence converges and has the same limit as the first.*

In the foregoing definitions and theorems the point sets $M_n$ have been made to depend on a single integer, $n$. It is obvious that they may equally well depend on variables which in a more general sense approach limits; cf. the definition of Function in Chap. III, § 3. It is enough for our present purposes to consider a single extension, namely, to an aggregate of point sets $M_{nq}$, where $n$ and $q$ are natural numbers, and where the earlier condition $m \leqq n$ is now replaced by the pair: $m \leqq n$, $l \leqq q$.

**§ 7. Addition of Irrationals.** The sum* of two numbers, $A = (a_1, a_2)$ and $B = (b_1, b_2)$, shall be defined as the number $C = (c_1, c_2)$:

$$A + B = C,$$

---

* We now drop the expression "first combination", meaning by "sum" the same thing.

where

$$c_1 = a_1 + b_1, \qquad c_2 = a_2 + b_2;$$

or

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1,\ a_2 + b_2).$$

It is understood that each $a_1$ is combined with each $b_1$ to yield the $c_1$; and similarly for $c_2$.

To justify this definition observe first that

$$c_1 < c_2,$$

since

$$a_1 < a_2, \qquad b_1 < b_2.$$

Secondly, to an arbitrary positive fraction, $\epsilon$, there correspond a $c_1'$ and a $c_2'$ such that

$$c_2' < c_1' + \epsilon .$$

For, to an arbitrary $\epsilon' > 0$ there correspond the relations:

$$a_2' < a_1' + \epsilon', \qquad b_2' < b_1' + \epsilon'.$$

Hence

$$a_2' + b_2' < a_1' + b_1' + 2\,\epsilon',$$

and it remains only to set

$$\epsilon' = \tfrac{1}{2}\,\epsilon, \qquad c_1' = a_1' + b_1', \qquad c_2' = a_2' + b_2'.$$

Finally, the $c_1$, $c_2$ exhaust all the fractions with possibly one exception.

Addition obeys the Commutative Law:

I. $$A + B = B + A;$$

and also the Associative Law:

II. $$A + (B + C) = (A + B) + C.$$

It is obvious that

$$A + 0 = A, \qquad 0 + A = A,$$

where $A$ is any number.

*Subtraction.* The inverse of addition is *subtraction.* The problem is to solve the equation:

$$A + X = B.$$

Consider first the case $B = 0$. It is obvious that one solution is:

$$A' = (-a_2, -a_1).$$

Thus

$$A + A' = 0.$$

Let $A''$ be an arbitrary solution:

$$A + A'' = 0.$$

Then

$$A' + A + A'' = A',$$

and since

$$A' + A = 0,$$

it follows that

$$A'' = A'.$$

Hence $A'$ is unique.

Proceeding now to the general case we find as a necessary condition:

$$A' + A + X = A' + B,$$

$$X = B + A'.$$

Thus $X$ can be no other number; and we see that, conversely, this number is a solution. Hence subtraction is always possible and unique.

Observe the inequalities:

i) $$|A + B| \leqq |A| + |B|,$$

ii) $$|\{|A| - |B|\}| \leqq |A + B|.$$

iii) If $$|A - X| \leqq H,$$

then $$A - H \leqq X \leqq A + H,$$

and conversely.

**§8. Limit of the Sum of Two Variables.** *Let* $M_1, M_2, \cdots$ *and* $N_1, N_2, \cdots$ *be two convergent sequences of point sets. Let* $X$ *be a point of* $M_n$ *and* $Y$ *a point of* $N_n$. *Form the sequence of point sets* $P_1, P_2, \cdots$, *where* $P_n$ *consists of the points* $X + Y$. *Then* $P_n$ *converges, and*

$$\lim_{n=\infty} P_n = \lim_{n=\infty} M_n + \lim_{n=\infty} N_n,$$

*or*

$$\lim (X + Y) = \lim X + \lim Y.$$

We wish to show that, to an arbitrary[*] $\epsilon > 0$, there corresponds an $m$ such that

$$|U + V - X - Y| < \epsilon, \qquad m \leqq n,$$

where $X$ and $Y$ are points of $M_n$, $N_n$, and $m \leqq n$.

We know that, to any $\epsilon' > 0$, there corresponds an $m$ such that

$$|U - X| < \epsilon', \qquad |V - Y| < \epsilon',$$

where $X$, $Y$ are points of $M_n$, $N_n$, and $m \leqq n$. Hence

$$|U + V - X - Y| < 2\,\epsilon'.$$

If, then, we choose $\epsilon' = \frac{1}{2}\,\epsilon$, the proof is complete.

We could equally well have defined an aggregate of point sets $P_{nq}$ made up of the points $X$ of $M_n$ and $Y$ of $N_q$, and shown in a similar manner that

$$\lim_{(n,\,q)=(\infty,\,\infty)} P_{nq} = \lim_{n=\infty} M_n + \lim_{q=\infty} N_q\,.$$

## §9. Multiplication of Irrationals.

The product of two real numbers, $A = (a_1, a_2)$ and $B = (b_1, b_2)$, shall be defined as follows. Let fractions be chosen at pleasure:

$$\epsilon_1 \geqq \epsilon_2 \geqq \cdots, \qquad \lim_{n=\infty} \epsilon_n = 0;$$

$$\eta_1 \geqq \eta_2 \geqq \cdots, \qquad \lim_{n=\infty} \eta_n = 0.$$

Let $a_p$ $(p = 1, 2)$ be any point of the $\epsilon_n$-neighborhood of $A$, § 6, and let $b_p$ $(q = 1, 2)$ be any point of the $\eta_n$-neighborhood of $B$. Let $\{x\} = M_n$ consist of the points

$$x = a_p\,b_q.$$

Then these points approach a limit,

$$U = (u_1, u_2),$$

as $n$ becomes infinite. This number is defined as the *product* of $A$ and $B$:

$$U = A\,B,$$

$$(a_1, a_2)\,(b_1, b_2) = \lim_{n=\infty} \{\, a_p\,b_q \,\}.$$

---

[*] Now that addition has been defined it is obviously henceforth immaterial whether we require $\epsilon$ to be a positive fraction or an arbitrary positive number.

For, let $a_p b_q$ lie in $M_m$ and let $a'_{p'}$, $b'_{q'}$, lie in any $M_n$, $m \leqq n$, and so, also, in $M_m$.
Since

$$| a'_{p'} - a_p | < 2\,\epsilon_m, \qquad | b'_{q'} - b_q | < 2\,\eta_m,$$

it follows that

$$a'_{p'} = a_p + \zeta_1, \qquad | \zeta_1 | < 2\,\epsilon_m,$$

$$b'_{q'} = b_q + \zeta_2, \qquad | \zeta_2 | < 2\,\eta_m.$$

Hence

$$a'_{p'}\, b'_{q'} - a_p b_q = b_q \zeta_1 + a_p \zeta_2 + \zeta_1 \zeta_2.$$

Now, there is a constant $G$ such that

$$| a_p | < G, \qquad | b_q | < G$$

for any $a_p$, $b_q$ under consideration. Consequently

$$| a'_{p'}\, b'_{q'} - a_p b_q | < G(2\,\epsilon_m + 2\,\eta_m) + 4\,\epsilon_m\,\eta_m.$$

If, now, a positive fraction $\epsilon$ be chosen at pleasure, $m$ can be so determined that

$$G(2\,\epsilon_m + 2\,\eta_m) + 4\,\epsilon_m\,\eta_m < \epsilon.$$

Consequently the $\epsilon$-neighborhood of a point $\xi$ of $M_m$—namely, $\xi = a_p b_q$—contains all the points of each later $M_n$ $(m < n)$ and so the sequence of point sets $M_n$ converges.

*Idemfactor.* There is one number of the system, namely:

$$I = (a_1, a_2), \qquad a_1 \leqq 1 < a_2,$$

which has the property that

$$IA = AI = A,$$

where $A$ is any number of the system; and $I$ is unique, as will appear from the uniqueness of division. $I$ is the number 1.

The truth of the *Commutative Law*:

I. $$AB = BA,$$

is obvious from the definition. The *Associative Law*:

II. $$(AB)C = A(BC),$$

is also true. For, let fractions

$$\lambda_n \geqq \lambda_{n+1}, \qquad \lim_{n=\infty} \lambda_n = 0,$$

be chosen arbitrarily, and let $c_r$ $(r = 1, 2)$ be a point of the $\lambda_n$-neighborhood of $C = (c_1, c_2)$. Let $M'_n$ consist of the points

$$x = a_p\, b_q\, c_r\,.$$

Then it is shown exactly as before that these points $x$ approach a limit,

$$V = \lim_{n=\infty} \{ a_p\, b_q\, c_r \}\,.$$

This number, $V$, has the value $UC$, where $U = AB$. For, the definition of the latter product is as follows. Let $U = (u_1, u_2)$. Let fractions

$$\mu_n \geqq \mu_{n+1}\,, \qquad \lim_{n=\infty} \mu_n = 0,$$

be chosen, and let $u_\pi$, $(\pi = 1, 2)$ be a fraction lying in the $\mu_n$-neighborhood of $U$. Let $c_r$, as before, lie in the $\lambda_n$-neighborhood of $C$. Then $N_n$ shall consist of the points $u_\pi\, c_r$, and

$$UC = \lim_{n=\infty} \{ u_\pi\, c_r \}\,.$$

Since the points $a_p\, b_q$ for a given $n$ include all the fractions lying in a certain neighborhood of the point $U = AB$, it is clear that the $\mu_n$ can be so chosen that the $\mu_n$-neighborhood of $U$ will include only such fractions $u_\pi$ as are contained among the above $a_p\, b_q$. When this is done, the points of the set $\{ u_\pi\, c_r \}$ for a given $n$ are all contained among the points of the set $\{ a_p\, b_q\, c_r \}$ corresponding to the same $n$. Thus the $\{ u_\pi\, c_r \}$ are sub-sets of the $\{ a_p\, b_q\, c_r \}$, and hence, by Theorem 2 of § 6, their limits are equal, or:

$$V = UC = (AB)\, C.$$

On the other hand, $V$ is invariant of the order of the factors, $a_p$, $b_q$, $c_r$. Hence

$$V = (BC)\, A,$$

and since the commutative law holds,

$$V = A\, (BC).$$

Thus the Associative Law is established.

*Division* is the inverse of multiplication and is expressed by the equation:

$$AX = B, \qquad A \neq 0.$$

First, let $B = I$, the idemfactor, $I = 1$:

$$AX = 1.$$

One solution of this equation is the following. Let a set of fractions be chosen:

$$\epsilon_1 \geqq \epsilon_2 \geqq \cdots, \qquad \lim_{n=\infty} \epsilon_n = 0,$$

and let $\epsilon_1 < | A |$. Let $A = (a_1, a_2)$, and let $M_n$ be the points $1/a_p$ $(p = 1, 2)$ where $a_p$ lies in the $\epsilon_n$-neighborhood of $A$. Then the sequence of point sets $M_1, M_2, \cdots$ converges. For, let $a_p$, $a'_q$ be any two points of $M_n$, and let $G$ be a positive fraction $< | A | - \epsilon_1$. Then

$$\frac{1}{a_p} - \frac{1}{a'_q} = \frac{a'_q - a_p}{a_p\, a'_q},$$

and hence

$$\left| \frac{1}{a_p} - \frac{1}{a'_q} \right| = \frac{| a' - a_p |}{| a_p\, a'_q |} < \frac{2\,\epsilon_n}{G^2}.$$

Denote the limit by $A'$:

$$AA' = A'A = 1.$$

It is called the *reciprocal* of $A$, and is unique. For if

$$AA'' = 1,$$

then $$A'AA'' = A',$$

or $$A'' = A'.$$

Turning now to the general equation,

$$A\,X = B,$$

multiply by $A'$:

$$A'A\,X = A'\,B,$$

$$X = BA'.$$

Thus division is always possible when $A \neq 0$. Moreover, it is unique. For, if

$$A\,X = A\,Y,$$

then $$A'A\,X = A'A\,Y,$$

and so $$X = Y.$$

Multiplication has been defined and the foregoing properties have been established, independently of addition — in fact, the whole treatment could precede addition. We turn now to the last of the five formal laws, which combines both addition and multiplication.

*The Distributive Law:*

$$A(B+C)=AB+AC.$$

Here, if $A=(a_1, a_2)$, $B=(b_1, b_2)$, and $C=(c_1, c_2)$, the left-hand side is, by definition:

$$A(B+C)=\lim_{n=\infty}\{a_p(b_q+c_q)\}.$$

On the right,

$$AB=\lim_{n=\infty}\{a_p b_q\}, \qquad AC=\lim_{n=\infty}\{a_{p'} c_{q'}\}.$$

By the theorem of § 7 and Theorem 2, § 6 the first of these three limits is equal to the sum of the last two, and this proves the theorem. Thus the five formal laws of algebra hold for irrationals.

*The System of Real Numbers.* By this is meant the class of numbers defined above as cuts, plus the four species defined for these numbers. This number system has the property that a product vanishes when and only when one of the factors vanishes.

**§ 10. Roots. Inequalities.** From the definition of a number as a cut follows at once that the equation

$$x^n=a, \qquad 0 \leqq a,$$

admits one and only one positive root (or 0):

$$x=\sqrt[n]{a}.$$

For, let $a_2$ be any positive fraction such that

$$a < a_2^n,$$

and let $a_1$ be any other fraction. Then

$$x=(a_1, a_2)$$

is a solution of the equation, and the only solution which is not negative.

*Inequalities.* Any two numbers of the system of real numbers are equal or unequal, and if $a<b$, then $b>a$. Moreover, if $a<b$ and $b<c$, then $a<c$. Between any two numbers, $a<b$, lies a number, $c$: $a<c<b$. Also, if $0<a$ and $0<b$, there is a natural number $n$ such that $na>b$.

The student will find it convenient to have firmly in mind the further relations:—

i) If $a < b$ and $c \leqq d$, then $a + c < b + d$.

ii) If $0 \leqq a < b$ and $0 \leqq c < d$, then $ac < bd$.

iii) If $a < b$, then $\begin{cases} ka < kb, & \text{if } 0 < k; \\ ka > kb, & \text{if } k < 0. \end{cases}$

iv) If $0 \leqq a < b$, then

$$\sqrt[n]{a} < \sqrt[n]{b} \quad \text{and} \quad a^n < b^n.$$

v) $$|a + b| \leqq |a| + |b|.$$

vi) $$|\{|a| - |b|\}| \leqq |a + b|.$$

vii) If $|a - x| \leqq h$, then

$$a - h \leqq x \leqq a + h,$$

and conversely.

**§ 11. Retrospect.** Starting with the natural numbers and the four species, we proceeded to the positive fractions, and thence to the negative fractions, thus obtaining the system of rational numbers.

From this point we introduced the irrational numbers by means of the cut. The definitions of equality and inequality, and of addition, were most natural and simple. Even before addition was defined, it was possible to formulate and prove the fundamental theorem concerning limits, and thus the basis for all analysis occupies the first place in the development of arithmetic. Multiplication derives its definition from this principle.

On the other hand one might reasonably seek to create the number system first. Addition is defined naturally by means of the definition of number as a cut:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2).$$

One would like to define multiplication in the same manner:

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2).$$

But this is impossible when we start with the system of (positive and negative) rational numbers, since the fractions $a_1 b_1$ and $a_2 b_2$ do not define a cut.

There is a way cut. Introduce the irrational numbers after the fractions; i. e. before the negative numbers have been defined. Then the above definition of multiplication is valid. We thus obtain the system of positive real numbers. The negative numbers can now be defined precisely as before, and we arrive at the system of real numbers.

What have we lost by this method? Nothing of a logical nature; nothing of simplicity in the small. But in the large there is a lack of unity in the definition of the real numbers. There was no lack of unity in the definition of the system of rational numbers (positive and negative fractions, and 0). And there is no lack of unity when we advance from these to the real numbers by the cut. There is no lack of unity when we introduce the fundamental principle of limits even before the definition of addition. And from here on the development leaves nothing to be desired in point of simplicity.

All this has, however, an apologetic sound. Our views are expressed in negatives. It is important to make clear the positive reasons which justify the course we have followed. These are:

I. *The Cut.* The phenomenon of the partition of all the numbers of the system before us into two classes, the $a_1$ and the $a_2$, with

$$a_1 < a_2,$$

appears again and again in the most elementary and important considerations one meets in practice. It is natural, then, to seize on this mathematical manifestation as the defining element in extending the number system to its final scope. In Physics, that definition is best which lies closest to the heart of the phenomena to be analyzed, and this principle has its counterpart in Mathematics.

II. *Limits.* The prime object of extending the number system is continuity and the existence of a limit. If it were not for these phenomena, we could get on very well with the system of rational numbers (positive and negative fractions, and 0). Since the notion of the limit is the cause for generalizing arithmetic, its place is naturally the earliest possible one — immediately after the definition of the new numbers. From the point of view of scientific perspective, then, this arrangement leaves nothing to be desired.

So we need make no apology to the beginner for leading him over a path that is hard for his tender feet. He is being trained in the most important methods of the science, and his efforts will be rewarded by the acquisition of insight and power in analysis.

*The Method of Regular Sequences.* It is possible to introduce irrationals by a method which is, perhaps, more convenient for the beginner. Suppose we have established the system of rational numbers. By a *regular sequence* is meant a set of these numbers, $a_1, a_2, a_3, \cdots$ having the property that, if $\epsilon$ be a positive fraction chosen at pleasure, there corresponds an integer $m$ such that

$$|a_n - a_{n'}| < \epsilon, \qquad m \leqq n, n'.$$

The regular sequence is now made the element of a new logical class, the real numbers. If, in particular, there is a rational number, $a$, such that, to an arbitrary positive fraction $\epsilon$ corresponds an integer $m$, for which

$$|a - a_n| < \epsilon, \qquad m \leqq n,$$

this regular sequence is identified with $a$; i.e. set equal by definition to $a$. If not, a new number is introduced. In either case, the element of the logical class is the *mark* which consists in the regular sequence.

Two regular sequences, $A = (a_1, a_2, \cdots)$ and $B = (b_1, b_2, \cdots)$, are defined as *equal* if to an arbitrary positive fraction $\epsilon$ corresponds a natural number $m$ such that

$$|a_n - b_n| < \epsilon, \qquad m \leqq n.$$

The number $A$ is defined as *less* than $B$,

$$A < B,$$

if there are a natural number $m$ and a positive fraction $h$ such that

$$a_n + h < b_n, \qquad m \leqq n.$$

By the *sum* of two numbers,

$$C = A + B,$$

is meant the regular sequence

$$C = (a_1 + b_1,\ a_2 + b_2, \cdots).$$

Subtraction is always possible and unique:

$$A - B = (a_1 - b_1, a_2 - b_2, \cdots).$$

The *product* of two numbers,

$$C = AB,$$

is defined as the regular sequence

$$C = (a_1 b_1, a_2 b_2, \cdots).$$

Division by any number but 0 is always possible and unique:

$$\frac{A}{B} = \left[\frac{a_m}{b_m}, \frac{a_{m+1}}{b_{m+1}}, \cdots\right].$$

The justification for each of these definitions; i.e. the requisite convergence proof, is simple.

Thus the System of Real Numbers — the definition of irrationals and the extension of the four species to the new numbers — is established. The method has the advantage of simplicity in detail. It is well for the student, after a first study of the method of Dedekind, to work it through in all detail. He will then return to the former method with increased power and greater zest.

The method of regular sequences is a middle-of-the-road method. It is an easy way to reach the mountain top. The traveller buys his ticket and takes the funicular. Many people prefer this mode of travel. But some like a stiff climb over rocks and across streams, and such an ascent has its advantages if the heart is good and the muscles are strong.

# Chapter III

## Point Sets. Limits. Continuity

**§1. Definitions.** By *point* is meant the expression $(x_1, \cdots, x_n)$. The numbers $x_k$ are called its *coordinates*. A point in space of $n$ dimensions may be thought of as corresponding to it. But this idea is introduced merely for the convenience of geometric intuition, — not as an essential part of the concept. Any collection of points, defined so as to be recognizable, is called a *point set.* Each point is called an *element* of the set.

*Examples.* When $n = 1$:

a) The proper fractions.
b) The continuum $a < x < b$.
c) The line segment $a \leqq x \leqq b$.
d) The incommensurable numbers between 0 and 1.
e) The natural numbers.
f) Their reciprocals, $1, \frac{1}{2}, \frac{1}{3}, \cdots$.
g) The numbers $1, \frac{1}{2}, \frac{1}{3}, \cdots$ and 0.
h) The positive numbers.
i) All the real numbers.
j) A finite set of points, like 0, 1.

When $n > 1$:

k) The open square, when $n = 2$:

$$-1 < x < 1, \qquad -1 < y < 1.$$

l) The closed square, when $n = 2$:

$$-1 \leqq x \leqq 1, \qquad -1 \leqq y \leqq 1.$$

m) The rational points of space: $(x_1, \cdots, x_n)$, where the $x_k$'s are all fractions.

n) The surface of the unit (hyper-) sphere:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

o) The interior of the unit (hyper-) sphere:

$$x_1^2 + x_2^2 + \cdots + x_n^2 < 1.$$

*Distance.* By the *distance* between the points $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ is meant the number:

$$D = \sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2}.$$

*Bounded.* A point set is *bounded* if the distances of its points from a fixed point are all less than some fixed number.

*Finite.* A *finite point set* is one consisting of a finite number of points.

*Open Set.* If to an arbitrary point $(a_1, \cdots, a_n)$ of the set there corresponds a positive number $h$, dependent in general on $(a)$, such that all points $(x_1, \cdots, x_n)$ for which

$$a_k - h < x_k < a_k + h, \qquad k = 1, \cdots, n,$$

belong to the set, the set is called an *open set.*

*Neighborhood.* By the *neighborhood* of a point is meant a connected open set including the given point. It is usually a simple region, like a square or a circle with the point as its centre.

*Cluster Point.* A point $A$ is called a *cluster point* or *point of condensation* of a set if an arbitrary neighborhood of $A$, no matter how restricted, contains points of the set distinct from $A$.

*Isolated, Discrete.* A point $A$ of a set is called *isolated* if within a suitably chosen neighborhood of $A$ there are no other points of the set. A point set is said to be *discrete* if all its points are isolated; cf. Examples e), f), j).

*Boundary Point.* A boundary point of a set is a point in every neighborhood of which there is a point of the set and also a point not belonging to the set. The point itself may or may not belong to the set.

*Closed Set.* A set is *closed* if it is bounded and contains all its boundary points.

*Standard Element.* By this is meant the point set $(x_1, \cdots, x_n)$, where

$$a_k - h_k < x_k < a_k + h_k, \qquad 0 < h_k, \qquad k = 1, \cdots, n.$$

The point $(a_1, \cdots, a_n)$ is called the *centre.* The boundary points of a standard element are those points $(y_1, \cdots, y_n)$ for which

$$a_k - h_k \leqq y_k \leqq a_k + h_k, \qquad k = 1, \cdots, n,$$

at least *one* lower sign holding. The closed set consisting of a standard element and all its boundary points shall be called a *standard box.*

*Connected.* An open set, $M$, is said to be *connected* if it has the following property. Let $A$ and $B$ be any two of its points. Then there shall exist* $n + 1$ points of the set, $P_0 = A$, $P_1, \cdots, P_{n-1}$,

$P_n = B$, and $n$ standard elements, $R_0, R_1, \cdots, R_{n-1}$, where the centre of $R_k$ is $P_k$ and $R_k$ lies in $M$, such that $R_k$ contains $P_{k+1}$, $k = 0, 1, \cdots, n-1$.

*Dense.* A point set is *everywhere dense* in an open set if each point of the latter is a cluster point of the former. It is *intrinsically dense* if each of its points is a cluster point.

*Perfect.* A set is *perfect* if it is closed and intrinsically dense. A closed interval, $a \leqq x \leqq b$, is an example of a perfect set. But a perfect set does not necessarily contain any open set whatever; cf. § 2.

## LINEAR POINT SETS

*Upper Bound. Upper Limit.* A linear point set, i.e. a set for which $n = 1$, is said to have an *upper bound* if there is a fixed number $A$ such that

$$x \leqq A,$$

where $x$ is any point of the set. Any larger number, $B > A$, is also an upper bound of the set.

If $A$ belongs to the set, or if there are points of the set in every neighborhood of $A$, then $A$ is the *upper limit* (or *least upper bound*) of the set.

When the upper limit is a point of the set, it is called the *maximum.*

For example, the number 2 is an upper bound of the proper fractions. The number 1 is their upper limit. But the set has no maximum.

It is obvious how *lower bound*, *lower limit* or *greatest lower bound*, and *minimum* should be defined.

THEOREM 1. *A linear point set which has an upper bound, has an upper limit.*

Let $a_1$ be a fraction, to the right of which lies a point of the set. Consider the totality of such fractions. Let $a_2$ be any fraction not an $a_1$. Then these two classes of fractions define a cut, $(a_1, a_2)$, and thus determine a point, $U = (a_1, a_2)$. This point is the upper limit of the given point set. For,

i) it is an upper bound. Suppose there were a point $\xi$ of the set, such that $\xi > U$. Then there would be a fraction, $\alpha$, between $U$ and $\xi$:

$$U < \alpha < \xi.$$

This fraction would by definition be an $a_1$:

$$\alpha = a_1.$$

Thus we have the contradiction of an $a_1 > U$.

ii) $U$ is the least upper bound. For, suppose $A < U$ were an upper bound. Then there would be a fraction $\beta$ between $A$ and $U$:

$$A < \beta < U.$$

But $\beta$ would by definition be an $a_2$, and so we should have the contradiction of an $a_2 < U$.

Similarly, a linear point set which has a lower bound, has a lower limit.

By a *finite interval* is meant a point set

$$a < x < b,$$

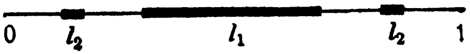
to which may be added one or both of the end points.

THEOREM 2. THE WEIERSTRASS-BOLZANO THEOREM. *An infinite linear point set which lies in a finite interval, has at least one cluster point.*

Consider the points $x$ of the line, such that, if $x'$ be any one of them, only a finite number of points of the given set (in particular, none whatever) lie to the left of $x$. This point set is bounded from above. It has, therefore, an upper limit, $G$. This point, $G$, is a point of condensation. For otherwise it would be an isolated point of the set, or not belong to the set at all. In either case, there would be a point $x$ to the right of $G$.

The theorem is true for a point set in space of $n$ dimensions. It can be proved conveniently by means of the Covering Theorem, § 11.

**§2. An Example.** Consider the interval $0 \leqq x \leqq 1$. From it we proceed to remove certain points. The point set we thus construct consists of the points that remain.



*Step* 1. From the middle of the interval remove the points of an interval of length

$$l_1 = \lambda - \frac{1}{3}\lambda,$$

where $0 < \lambda \leqq 1$ is a constant chosen once for all. More precisely, the points removed are those for which

$$\frac{1}{2} - \frac{1}{2} l_1 < x < \frac{1}{2} + \frac{1}{2} l_1.$$

*Step* 2. From the middle of each of the remaining intervals remove an interval whose length, $l_2$, is such that

$$l_1 + 2\, l_2 = \lambda - \frac{1}{4}\lambda$$

More precisely, the points removed are those for which

$$a - \frac{1}{2} l_2 < x < a + \frac{1}{2} l_2,$$

where $a$ denotes successively the mid-point of each interval remaining after the first step.

*n-th Step.* From the middle of each of the intervals remaining after the $(n - 1)$-st step remove an interval of length $l_n$, where

$$l_1 + 2\, l_2 + 2^2 l_3 + \cdots + 2^{n\;1} l_n = \lambda - \frac{1}{n+2}\lambda.$$

Those points which remain, no matter how large $n$ be taken, constitute the set we undertook to construct.

The set is perfect, for it is closed, and each of its points is a cluster point. It is, however, dense in no interval, for an arbitrary interval contains intervals which have been removed.

The sum of the intervals removed up to the $(n + 1)$-st step approaches a limit, namely, $\lambda$, as $n$ increases.

The points of the set can be enclosed in a finite set of intervals, the sum of whose lengths, however, will exceed the value $1 - \lambda$. If $\lambda = 1$, this sum can be made arbitrarily small. But if $\lambda < 1$, this is not the case.

Curiously enough, the points of the set admit a simple arithmetic expression when $\lambda = 1$. They are the numbers

$$x = 0.\,\alpha_1\,\alpha_2\,\alpha_3 \cdots = \frac{\alpha_1}{3} + \frac{\alpha_2}{3^2} + \frac{\alpha_3}{3^3} + \cdots$$

expressed in the triadic system, where the $\alpha$'s take on at pleasure the values 0, 2.

Incidentally, these points can be transformed on the points of the closed interval, but not in a one-to-one manner. Let

$$\beta_n = \frac{1}{2}\alpha_n,$$

and consider the number $\gamma$ expressed in the diadic system as

$$\gamma = 0.\beta_1\beta_2\beta_3\cdots = \frac{\beta_1}{2} + \frac{\beta_2}{2^2} + \frac{\beta_3}{2^3} + \cdots .$$

It is obvious that every number $\gamma$ of the interval

$$0 \leqq \gamma \leqq 1$$

is thus obtained at least once, but some points are obtained twice. Thus we have a transformation of the points of a closed interval on the points of a perfect set which is nowhere dense in any interval, some points of the former set even going over into two points of the latter.

§ 3. **Functions.** Let an arbitrary point set $M$ in space of $n$ dimensions be given. Let $(x_1, \cdots, x_n)$ be any one of its points, $P$. Let a number $y$ be assigned to $P$ by any law whatever. Then $y$ is called a *function* of $P$:

1) $$y = f(x_1, \cdots, x_n).$$

Represent $y$ as a point of the linear manifold, $N_1$. Then the law whereby the function is defined establishes a *transformation*[*] of the points of $M$ on the points of $N_1$. This transformation can be represented by the aid of the surface (i.e. hypersurface, manifold). defined by 1) in space of $n+1$ dimensions. Call it $\Sigma$. In the simplest case, $n = 1$, $\Sigma$ is a curve, provided that $M$ is an interval. But it is not this manifold *apart* that expresses the idea of the function. It is only when we relate it to the manifold $M$ and think of it as the collection of points which have gone out of $M$ by the given transformation, that we reproduce the entire conception of the *function*. Thus when $n = 1$ we may draw the picture of the set of vectors parallel to the $y$-axis, the initial point lying in $(x, 0)$ and the terminal point in $(x, y)$. This manifold of vectors, localized as stated, does give a complete account of the conception of the function. The locus of their terminal points is the manifold $\Sigma$. But $\Sigma$ by itself does not yield the conception of the function. It is only when we think of $\Sigma$ as defining the set of vectors, that we arrive at the basal idea of function.

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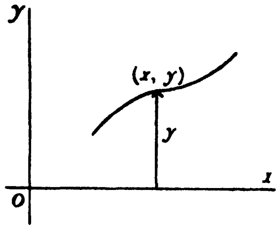
[*] Carathéodory begins his account of the conception of a function with the words: *Der moderne Begriff einer Funktion deckt sich mit dem einer Zuordnung.* Cf. his *Vorlesungen über reelle Funktionen*, p. 71, § 83.



On the other hand, since the manifold $\Sigma$ does determine uniquely the set of vectors, it yields a means of defining or representing the function. The function is the set of vectors. The manifold $\Sigma$ stands in a one-to-one relation to the concept: *function*, and for many purposes can be used to represent the function, with the consciousness of the vectors.

We have spoken only of the case of single-valued functions. The extension to multiple-valued functions is obvious.

At the beginning, we spoke of $y$ as the *function*. Thus we should say: The value of the function

$$y = x^2 + 1,$$

when $x = 1$, is 2. This is a different meaning of the term. Both meanings exist side by side in common parlance, but no confusion of ideas arises from these two uses of the word.

**§ 4. Continuation.** In the last paragraph we thought of the domain of definition of the function as a point set; and we transformed each point $P$ of the set on a point $y$ of the linear manifold. We may express this idea trenchantly by saying: $y$ is a *point function*, and write:

$$1)\qquad y = f(P).$$

We may generalize this idea by beginning with a *set of manifolds*. For example, we might take an arc of a curve, like an arc of a circle, or an arc of a continuous curve that has no tangent (Chap. IV, § 2) or an arc of a simple Jordan curve (Chap. VII, § 10) and then inscribe a broken line. These broken lines would serve as an illustration of such a set of manifolds as we have in mind We might attach to each element (i. e. each broken line of the set) a number, namely, its length, $y$. Thus a transformation of the elements of this set of manifolds on the points of a linear manifold would be defined, and this transformation gives us an extended idea of the concept *function*. If we let $M$ refer to any one of the broken lines, we could write suggestively

$$2) \qquad y = f(M),$$

and call $y$ a *function* of $M$.

Again, consider an arbitrary bounded point set $S$ in the plane. Let $M$ be a polygon, or a number of polygons, each set $M$ containing the points of $S$ in its interior, and each member of a given set $M$ being bounded by lines parallel to the axes. More precisely, these lines shall be chosen from the lines

$$3) \qquad x = \frac{p}{2^n}, \qquad y = \frac{q}{2^n},$$

where $n$ is a natural number and $p, q$ are whole numbers. To each of these sets $M$ we could attach the number $y$ which represents its total area*, and thus each $M$ would be transformed on a point of a linear point set. This transformation may be taken as an example of the extended concept *function*.

Another example is given by removing the requirement that the sides of the polygons lie along the lines 3), and allowing them to be any parallels to the axes.

These examples illustrate the generalization of the concept *function*, which we will now formulate as follows. Consider a collection of point sets $A$. Each $A$ may be any point set one pleases to define in space of $n$ dimensions. The totality of these $A$'s shall be referred to as the manifold, $\mathfrak{A}$, of the $A$'s. To each $A$ shall be attached a number, $y$. Thus the elements $A$ of $\mathfrak{A}$ are transformed on the points

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* For purposes of illustration we could also take as $y$ the total perimeter, or the sum of all the diagonals.

On the other hand, $S$ might be an arbitrary manifold in space, and $M$, a polyhedron containing $S$; its sides, or faces, lying in the planes:

$$x = \frac{p}{2^n}, \qquad y = \frac{q}{2^n}, \qquad z = \frac{r}{2^n}.$$

The extension to space of any number of dimensions is now obvious.

of a line. This transformation is a concept which we denote as a *function*. We write:

$$y = f(A),$$

and say that $y$ is a *function* of $A$. The manifold $\mathfrak{A}$ is the domain of the independent variable, $A$; or the domain of definition of the function.

A third extension of the concept *function* is the following. We start, as in the last case, with a manifold $\mathfrak{A}$ of point sets $A$, and now we assign to each $A$ a point set $B$ according to any law whatever. Denote the manifold of the $B$ by $\mathfrak{B}$. Then this transformation of $\mathfrak{A}$ on $\mathfrak{B}$ through $A \longrightarrow B$ shall also define the concept *function*. Thus in polar reciprocation a plane determines a point.

§ 5. **Limits.** We begin with the simplest case of a function,

1) $$y = f(x),$$

defined in the points $x$ of a linear set $A$. Let $a$ be a cluster point of $A$.

By the *$\delta$-neighborhood* of the point $a$ we mean the point set

$(\delta)_a$: $$a - \delta < x < a + \delta,$$

or, what is the same thing, the points $x$ for which

$$|x - a| < \delta.$$

By the *abridged $\delta$-neighborhood* of $a$ we mean the point set $(\delta)'_a$ obtained from $(\delta)_a$ by removing the point $a$; i.e. the point set

$(\delta)'_a$: $$0 < |x - a| < \delta.$$

*Definition.* The function $f(x)$ shall be said to *approach a limit* when $x$ approaches $a$ if to an arbitrary positive $\epsilon$ there corresponds a $\delta$ such that

$$|f(x') - f(x'')| < \epsilon,$$

where $x'$, $x''$ are any two points of $(\delta)'_a$ which lie in $A$.

According to the theorem of Chap. II, § 6 there then exists a number $b$ such that, to an arbitrary $\epsilon$-neighborhood of $b$,

$(\epsilon)_b$: $$|y - b| < \epsilon,$$

there corresponds an abridged $\delta$-neighborhood of $a$, $(\delta)'_a$, with the property that the value $y$ of the function for an arbitrary $x$ of $A$ lying in $(\delta)'_a$ is a point of $(\epsilon)_b$, or:

$$|b-f(x)| < \epsilon, \qquad 0 < |x-a| < \delta,$$

where $x$ lies in $A$.

Conversely, if $f(x)$ has the latter property, then it approaches a limit.

The number $b$ is called the *limit*, and we write:

2) $$\lim_{x=a} f(x) = b.$$

*Relation to the Earlier Definition.* The relation of the present definition of a *function's approaching a limit* to the earlier definition of the *limit of a point set* is as follows. Here, we start with a variable point set, namely, the points $\{x\}$ of $A$ which lie in the abridged $\delta$-neighborhood of $a$, i.e. in $(\delta)'_a$. This point set, $\{x\}$, approaches a limit, namely, $a$, as $\delta$ approaches 0, by the definition of Chap. II, § 6.

By means of the function 1) this point set $\{x\}$ is transformed into a point set $\{y\}$, depending on $\delta$. And now the function 1) is said to *approach a limit* if this latter point set, $\{y\}$, approaches a limit in the earlier sense.

*Notation.* This symbol, Equation 2), shall have the following meaning: i) it asserts that $f(x)$ approaches a limit as $x$ approaches $a$; i.e. it vouches for the *existence* of a limit; and ii) it asserts that the value of the limit is the number $b$.

THEOREM. *Let $A_1$ be a sub-set of $A$ having $a$ as a cluster point. If $f(x)$, regarded as a function whose domain of definition is $A$, approaches a limit, when $x$ approaches $a$, then $f(x)$, regarded as a function whose domain of definition is $A_1$, will also approach a limit, when $x$ approaches $a$, and the two limits will be equal. But the converse is not true.*

*Unilateral Approach.* If the points of $A$ near $a$ lie above $a$: i.e. if $x \geqq a$, and if $f(x)$ approaches a limit, we may write:

3) $$\lim_{x=a+} f(x);$$

with a similar definition for

4) $$\lim_{x=a-} f(x).$$

If, on the other hand, $f(x) \geqq b$, we may write:

5)
$$\lim_{x=a} f(x) = b^+.$$

Similarly, if $f(x) \leqq b$, we may write:

6)
$$\lim_{x=a} f(x) = b^-.$$

Of course, definitions 5) and 6) can be combined in all possible ways with 3) and 4).

*Becoming Infinite.* The function is said to become *positively infinite*:

$$\lim_{x=a} f(x) = +\infty,$$

if to a number $G$ chosen arbitrarily large there corresponds a $\delta$ such that

$$G < f(x),$$

where $x$ is any point of $(\delta)'_a$ which lies in $A$.

The corresponding definition for becoming negatively infinite,

$$\lim_{x=a} f(x) = -\infty,$$

is obvious.

We say, $f(x)$ *becomes infinite*:

$$\lim_{x=a} f(x) = \infty,$$

if $|f(x)|$ becomes positively infinite.

A necessary and sufficient condition that $f(x)$ become infinite, or positively infinite, or negatively infinite, is that

$$\lim_{x=a} \frac{1}{f(x)} = 0, \qquad \text{or} \qquad \lim_{x=a} \frac{1}{f(x)} = 0^+ \qquad \text{or} \qquad \lim_{x=a} \frac{1}{f(x)} = 0^-.$$

The independent variable $x$ may of course approach $a$ unilaterally.

*Poles.* When a function $f(x)$ becomes infinite at a point $a$, it is customary to say that the function has a *pole* at $a$, and to write:

$$f(a) = \infty.$$

No statement is thereby made as to whether the function is defined in the point $a$, or if it is, what its value there is. Thus if

$$\begin{cases} f(x) = \dfrac{1}{x}, & x \neq 0; \\ f(0) = 0, & \end{cases}$$

it is true that

$$f(0) = \infty \qquad \text{and} \qquad f(0) = 0.$$

The last equation states the value of the function when $x = 0$. The first equation has no concern with the value of the function when $x = 0$. It makes a certain assertion about the function when $x \neq 0$.

*Independent Variable Becoming Infinite.* Hitherto we have considered only the case that the domain of definition $A$ of the function $f(x)$ has a cluster point $x = a$. If $A$ has no upper bound, we say that $f(x)$ *approaches a limit* when $x$ *becomes infinite* (more precisely, *positively infinite*) if to an arbitrary positive $\epsilon$ there corresponds a number $G$ such that

$$| f(x') - f(x'') | < \epsilon$$

for any two points $x'$, $x''$ of $A$, lying in the interval

$$G < x.$$

This interval is the analogue of the abridged $\delta$-neighborhood $(\delta)'_a$ of the point $x = a$. There then exists a number $b$ as before, such that

$$| b - f(x) | < \epsilon, \qquad G' < x;$$

and we write:

$$\lim_{x=\infty} f(x) = b$$

or, more precisely:

$$\lim_{x=+\infty} f(x) = b.$$

The definition

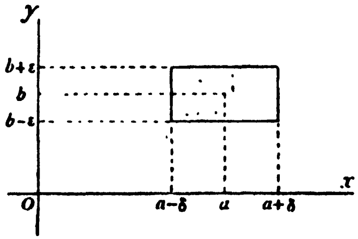
$$\lim_{x=-\infty} f(x) = b$$

is now obvious; as are also the definitions

$$\lim_{x=\infty} f(x) = b^{+}, \quad \text{etc.}$$

### EXERCISE

Illustrate each of these definitions by a suitable figure, carefully defined, such as is suggested by the diagram.



### EXERCISES*

1. Show that the function

$$y = x \sin \frac{1}{x}$$

approaches the limit 0 when $x$ approaches 0.

2. Show that

$$\lim_{x=0^+} \frac{1}{1+e^{\frac{1}{x}}} = 0, \qquad \lim_{x=0^-} \frac{1}{1+e^{\frac{1}{x}}} = 1.$$

3. Does the function

$$y = x \sin x$$

become infinite when $x = \infty$?

4. Show that the function

$$y = x + 2 \sin x$$

becomes infinite when $x = \infty$.

Does this function steadily increase as $x$ increases?

5. If $f(x)$ approaches a limit as $x$ approaches $a$, and if

$$A < f(x) < B,$$

show that

$$A \leqq \lim f(x) \leqq B.$$

6. Let $f(x)$ approach a limit when $x$ approaches $a$, and let there be no $(\delta)_a'$ in which $f(x)$ is constant. To a given $\epsilon$ correspond infinitely many values of $\delta$. Show that, for any given $\epsilon$ below a certain positive constant: $\epsilon < h$, these $\delta$'s form a bounded set, and this set has a maximum.

**§6. Bounded Functions.** A function $f(x_1, \cdots, x_n)$ is said to be *bounded at the point* $P$ if every neighborhood of $P$ contains a point of the domain of definition $A$ of the function and furthermore there exists a certain neghborhood $\mathfrak{U}$ of $P$ and a positive constant $M$ such that

$$|f(x_1, \cdots, x_n)| < M$$

for all points of $A$ which lie in $\mathfrak{U}$.

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[*] In these Exercises, which serve solely as illustrations, the properties of the elementary functions are assumed. A systematic developement of these functions will presently be given.

Thus the function

$$1) \qquad y = \sin\frac{1}{x}$$

is bounded at the point $x = 0$. Again, the function

$$2) \qquad y = \frac{1}{x}, \qquad 0 < x \leqq 1$$

is bounded at every point of its domain of definition.

A function is said to be *bounded in a region* if the region contains a point of $A$, and if there exists a positive contant $M$ such that

$$|f(x_1, \cdots, x_n)| < M$$

for all points of $A$ which lie in the region.

Thus the function 2) is bounded in every point of its domain of definition, but it is not bounded in that domain.

A function may be bounded at a point, as

$$y = \sin\frac{1}{x}$$

at the point $x = 0$, or it may have a pole at the point, as

$$y = \frac{1}{x}$$

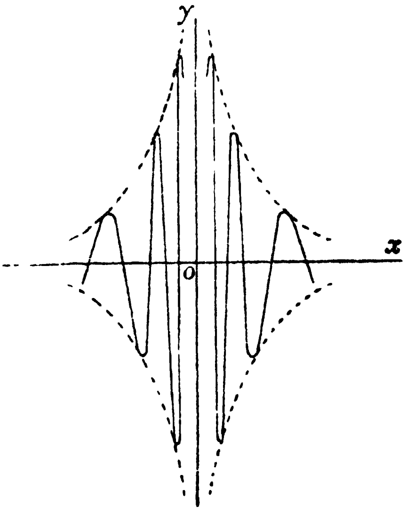
at the point $x = 0$. But these two cases do not exhaust all the possibilities. Consider the function

$$3) \qquad y = \frac{\sin\frac{1}{x}}{x}, \qquad x \neq 0.$$

This function is obviously not bounded at the point $x = 0$, for $\sin\frac{1}{x} = 1$ when $x = 1/(\frac{\pi}{2} + 2k\pi)$, and the function has the value

$$\frac{\pi}{2} + 2k\pi.$$

But the function does



not have a pole at the point $x = 0$. For, $\sin\frac{1}{x} = 0$ when $x = 1/k\pi$, and so the function vanishes there. Hence it does not become infinite when $x$ approaches 0.

We see, then, that the behavior of a function at a point involves a *three-fold classification*:

i) The function may be bounded at the point;
ii) It may become infinite there; or
iii) It may neither be bounded nor become infinite there.

*Oscillation.* Let a function be bounded in a region. Let $G$ be its upper limit and $K$, its lower limit. The difference:

$$D = G - K,$$

is defined as the *oscillation* of the function *in the region.*

Let a function be bounded at a point $a$, and let $\delta$ be so chosen that the function is bounded in $(\delta)_a$. The lower limit of the oscillation $D$ for such regions $(\delta)_a$ is defined as the *oscillation* of the function *in the point a.*

**§ 7. Three Theorems on Limits.** THEOREM I. *If each of two variables, $X$ and $Y$, approaches a limit, their sum approaches a limit, and the limit of their sum is equal to the sum of the limits:*

$$\lim (X + Y) = \lim X + \lim Y.$$

We will prove the theorem for the case of functions of a single variable $x$ with a cluster point $x = a$ of the domain of definition, $M$. The proof admits of immediate generalization to all the higher cases that arise in point functions.

Let the limits of $X$ and $Y$ be $A$ and $B$. Let

$$X = A + \zeta, \qquad Y = B + \eta.$$

Then $\zeta$ and $\eta$ are infinitesimals; i.e. variables which approach the the limit 0. More precisely, to an arbitrary $\epsilon > 0$ there corresponds a positive $\delta$ such that

1) $\quad |\zeta| < \epsilon, \qquad 0 < |x - a| < \delta, \qquad x$ in $M$.

Similarly,

2) $\quad |\eta| < \epsilon, \qquad 0 < |x - a| < \delta', \qquad x$ in $M$.

We may, however, without loss of generality drop the prime, setting $\delta' = \delta$, because a given $\delta$ can always be replaced by any smaller positive $\delta$.

We wish to prove that, to a positive $\epsilon$ chosen at pleasure there corresponds a positive $\delta$ such that

$$3)\qquad |A+B-(X+Y)|<\epsilon,\qquad 0<|x-a|<\delta,\qquad x \text{ in } M.$$

Now the $\epsilon$'s that appear here — the $\epsilon$ of 1) and 2), and the $\epsilon$ of 3) — are of different origins. The first $\epsilon$ (call it $\epsilon'$) is a number at our disposal. We can choose it to suit our purposes. But the second $\epsilon$ pertains to the conclusion. It is given us, as it were, by our adversary, and we have to meet it, — to find a $\delta$ that will match it and make 3) a true relation. In other words, we have to prove that to the $\epsilon$ of our adversary corresponds a positive $\delta$ such that 3) is true, or:

$$4)\qquad |\zeta+\eta|<\epsilon,\qquad 0<|x-a|<\delta,$$

it being understood henceforth that $x$ lies in $M$.

To do this we infer from 1) and 2) that

$$|\zeta|+|\eta|<2\epsilon',\qquad 0<|x-a|<\delta.$$

Furthermore,

$$|\zeta+\eta|\leqq|\zeta|+|\eta|.$$

Hence

$$|\zeta+\eta|<2\epsilon',\qquad 0<|x-a|<\delta.$$

If, then, we choose our $\epsilon'$ so that $2\epsilon'\leqq\epsilon$, the relation 4) will be fulfilled, and thus the theorem is proved.

THEOREM II. *If each of two variables, $X$ and $Y$, approaches a limit, their product approaches a limit, and the limit of their product is equal to the product of their limits:*

$$\lim(XY)=[\lim X][\lim Y].$$

We wish to prove that to an arbitrary positive $\epsilon$ corresponds a positive $\delta$ such that

$$5)\qquad |AB-XY|<\epsilon,\qquad 0<|x-a|<\delta.$$

And we know that, when we decide on an $\epsilon'>0$, we can find a $\delta>0$ such that

$$|\zeta|<\epsilon',\qquad |\eta|<\epsilon',\qquad 0<|x-a|<\delta.$$

Now,

$$XY=AB+B\zeta+A\eta+\zeta\eta.$$

Hence

$$|AB - XY| = |B\zeta + A\eta + \zeta\eta|$$

$$\leqq |B|\,|\zeta| + |A|\,|\eta| + |\zeta|\,|\eta| \leqq [\,|A| + |B| + \epsilon'\,]\,\epsilon'.$$

It remains merely to show that we can choose our $\epsilon'$ so that this last expression is less than our opponent's $\epsilon$. This can be done conveniently by first restricting $\epsilon'$ to be $< 1$, and then restricting $\epsilon'$ further by the condition:

$$(\,|A| + |B| + 1\,)\,\epsilon' \leqq \epsilon.$$

Thus the proof is complete.

THEOREM III. *If each of two variables, X and Y, approaches a limit, and if* $\lim Y \neq 0$, *then the quotient X/Y approaches a limit, and the limit of the quotient is the quotient of the limits:*

$$\lim\left(\frac{X}{Y}\right) = \frac{\lim X}{\lim Y}.$$

The relation which here corresponds to 3) and 5) above, is:

$$6)\qquad \left|\frac{A}{B} - \frac{X}{Y}\right| < \epsilon, \qquad 0 < |x - a| < \delta.$$

Now,

$$\frac{A}{B} - \frac{X}{Y} = \frac{A}{B} - \frac{A + \zeta}{B + \eta} = \frac{A\eta - B\zeta}{BY}.$$

First choose $\delta$ so that

$$|Y| > \tfrac{1}{2}\,|B|$$

Then

$$\left|\frac{A}{B} - \frac{X}{Y}\right| \leqq 2\,\frac{|A|\cdot|\eta| + |B|\cdot|\zeta|}{B^2},$$

and the remainder of the proof presents no difficulties.

## EXERCISES

1. If $X$ approaches a limit, and if $C$ is a constant, then

$$\lim(X + C) = \lim X + C;$$

$$\lim(CX) = C\lim X;$$

$$\lim\frac{C}{X} = \frac{C}{\lim X}, \text{ provided } \lim X \neq 0;$$

$$\lim X^n = (\lim X)^n.$$

2. If $X_1, \cdots, X_n$ are variables, each of which approaches a limit, and if $C_1, \cdots, C_n$ are constants, then

$$\lim (C_1 X_1 + \cdots + C_n X_n) = C_1 \lim X_1 + \cdots + C_n \lim X_n.$$

3. If, furthermore, $E_1, \cdots, E_n$ are constants, then

$$\lim \frac{C_1 X_1 + \cdots + C_n X_n}{E_1 X_1 + \cdots + E_n X_n} = \frac{C_1 \lim X_1 + \cdots + C_n \lim X_n}{E_1 \lim X_1 + \cdots + E_n \lim X_n},$$

provided

$$E_1 \lim X_1 + \cdots + E_n \lim X_n \neq 0.$$

4. Show that, if $f(x)$ is a function which is bounded at the point $x = a$, and if

$$\lim_{x=a} \varphi(x) = 0,$$

then

$$\lim_{x=a} f(x)\, \varphi(x) = 0.$$

5. If $G(x)$ is a polynomial in $x$:

$$G(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_n;$$

and if $X$ is a variable which approaches a limit, then $G(X)$ approaches a limit, and

$$\lim\ G(X) = c_0\ (\lim X)^n + c_1\ (\lim X)^{n-1} + \cdots + c_n.$$

6. State and prove an analogous theorem relating to the quotient of two functions:

$$\frac{\varphi(x)}{f(x)}.$$

7. If each of the variables $X_1, X_2, \cdots, X_n$ approaches a limit, and if $F(x_1, \cdots, x_n)$ is a polynomial, then

$$F(X_1, \cdots, X_n)$$

approaches a limit.

8. In only one of the foregoing questions is an $\epsilon$-proof required. Which one is it?

9. Each of Questions 1 — 7 contains one or more *existence theorems.* Did you prove the *convergence*, i. e. the *existence of the limit*, each time?

**§ 8. Continuous Functions.** Let $f(x)$ be a function defined in the points of a set $A$, and let $x = a$ be a cluster point of $A$. The function is said to be *continuous in a* if i) $f(x)$ approaches a limit when $x$ approaches $a$; ii) the value of this limit is $f(a)$:

$$\lim_{x=a} f(x) = f(a).$$

By § 4 this means that, to a positive $\epsilon$ chosen at pleasure, there corresponds a $\delta > 0$ such that

$$|f(a) - f(x)| < \epsilon, \qquad |x - a| < \delta, \qquad x \text{ in } A.$$

The function is said to be *continuous in A* if it is continuous in every point of $A$.

The region $A$ will usually consist of an interval, open or closed, finite or infinite. By a *closed interval* $(a, b)$ is meant, we recall, the set of points

$$a \leqq x \leqq b.$$

The definition may be illustrated geometrically as follows, Fig., p. 73. Plot the point $(x, y) = (a, b)$, where $b = f(a)$. Draw the horizontal parallels

$$y = b + \epsilon, \qquad y = b - \epsilon.$$

The point $(a, b)$ lies within the strip bounded by these lines. And now the definition says that there is a vertical strip bounded by the lines

$$x = a + \delta, \qquad x = a - \delta$$

such that, when $x$ lies between $a - \delta$ and $a + \delta$, the corresponding point $(x, y)$ lies in the rectangle common to these two strips.

*Example* 1. Consider the function:

$$f(x) = x, \qquad \text{when } x \text{ is rational;}$$
$$f(x) = -x, \quad \text{when } x \text{ is irrational.}$$

This function is continuous for $x = 0$, but for no other value of $x$.

*Example* 2. The function:

$$f(x) = 1, \quad x \text{ irrational;}$$
$$f(x) = 0, \quad x \text{ rational,}$$

is bounded. It is discontinuous for all values of $x$.

*Example* 3. Let $f(x)$ be defined in the interval $0 \leqq x \leqq 1$ as follows:

i) $$f(0)=f(1)=1\,;$$

ii) $$f\left(\frac{1}{2!}\right)=\frac{1}{2!}$$

iii) $$f\left(\frac{p}{3!}\right)=\frac{1}{3!}, \qquad p=1,2,4,5\,;$$

. . . . . . . . . . . . . . . . . . . . . . . .

n) $$f\left(\frac{p}{n!}\right)=\frac{1}{n!},$$

where $p/n!$ is a point of the interval, in which $f(x)$ has not yet been defined.

Finally, $f(x)=0$, $x$ irrational.

This function is discontinuous for all rational values of $x$, and continuous for all irrational values.

*Example* 4. The function $\varphi(x)$ obtained from the function $f(x)$ of the preceding example by periodicity:

$$\varphi(x)=f(x), \qquad 0<x\leqq 1\,;$$
$$\varphi(x+1)=\varphi(x), \qquad -\infty<x<\infty.$$

THEOREM 1. *If $f(x)$ is continuous in the point $x=\xi$, then $f(x)$ is bounded in this point.*

*If $f(\xi)\neq 0$, then $f(x)$ does not change sign or vanish in the neighborhood of $x=\xi$. More precisely:*

*If $f(\xi)>c$, then*

$$c<f(x)$$

*in a certain neighborhood of $\xi$.*

*If $f(\xi)<c$, then*

$$f(x)<c$$

*in a certain neighborhood of $\xi$.*

The proof follows from the fact that

$$f(\xi)-\epsilon<f(x)<f(\xi)+\epsilon, \qquad |x-\xi|<\delta, \qquad x \text{ in } A,$$

on setting $\epsilon=|f(\xi)-c|$

THEOREM 2. *A continuous function of a continuous function is a continuous function.*

*More precisely, let $y=\varphi(x)$ be continuous in a given point set $A$, and let*

$$u = f(y)$$

*be continuous in a point set B. Let* $\{y\}$ *be the point set defined by the values of* $\varphi(x)$. *Let B contain* $\{y\}$. *Then the function*

$$u = f[\varphi(x)]$$

*is continuous in A.*

The proof is left to the reader.

### Exercises

1. If $f(x)$ and $\varphi(x)$ are continuous in the point $x = a$ or in the region $A$, then

i) $$f(x) + \varphi(x),$$

ii) $$f(x)\ \varphi(x),$$

are continuous there. Moreover,

iii) $$\frac{f(x)}{\varphi(x)}$$

is continuous in $a$ or in a point $\xi$ of $A$, provided $\varphi(a) \neq 0$ or $\varphi(\xi) \neq 0$.

2. The function

$$y = x$$

is continuous for all values of $x$. Give a careful $\epsilon$-proof of this theorem.

3. The function

$$y = x^n, \qquad n = 2, 3, 4, \cdots,$$

is continuous for all values of $x$. Prove this theorem without the use of $\epsilon$'s, employing only the theorems of Question 2 and § 7.

4. Every polynomial in $x$ is continuous for all values of $x$.

5. A rational function

$$R(x) = \frac{f(x)}{\varphi(x)},$$

where $f(x)$ and $\varphi(x) \neq 0$ are any polynomials in $x$, is continuous for all values of $x$ for which $\varphi(x) \neq 0$.

6. If a function is continuous in a closed interval and vanishes there, show that its roots form a closed point set.

Is the theorem true if the interval is bounded, but not closed? Give an example.

7. Plot the graph of the function:

$$f(x) = x \sin\frac{1}{x}, \qquad x \neq 0;$$

$$f(0) = 0.$$

Prove that the function is continuous for all values of $x$. Assume that $\sin x$ is continuous and bounded.

## §9. Three Theorems on Continuous Functions.

THEOREM 1. *A function $f(x)$, continuous in a closed interval,*

$$a \leqq x \leqq b,$$

*is bounded.*

Assume the theorem false. Divide the given interval, $(a, b)$, into two equal subintervals. Then the theorem must be false for at least one of these. Let

$$\alpha_1 \leqq x \leqq \beta_1$$

be such an interval. If there is a choice, take, for definiteness, the left-hand interval.

Repeat the reasoning, subdividing $(\alpha_1, \beta_1)$, and denoting a subinterval in which the theorem is false by

$$\alpha_2 \leqq x \leqq \beta_2.$$

If there is a choice, take, for definiteness, the left-hand interval.

Proceeding in this manner we obtain a succession of numbers $\alpha_n$:

$$\alpha_1 \leqq \alpha_2 \leqq \cdots, \qquad \alpha_n < \beta_1;$$

and a second succession:

$$\beta_1 \geqq \beta_2 \geqq \cdots, \qquad \beta_n > \alpha_1.$$

Moreover,

$$\beta_n - \alpha_n \leqq \frac{b-a}{2^m}, \qquad m \leqq n.$$

These numbers determine a point $\xi$ such that

$$\alpha_n \leqq \xi \leqq \beta_n.$$

Since $f(x)$ is continuous in $x = \xi$, we have:

$$|f(\xi) - f(x)| < \epsilon, \qquad |x - \xi| < \delta,$$

or

$$|f(x)| < |f(\xi)| + \epsilon, \qquad |x - \xi| < \delta.$$

Let $m$ be so chosen that

$$\beta_m - \alpha_m < \delta.$$

Then the interval $(\alpha_m, \beta_m)$ lies within the interval $(\xi - \delta, \xi + \delta)$, and here is a contradiction; for in no interval $(\alpha_n, \beta_n)$ is the function bounded. This proves the theorem.

*Remark.* Observe that the theorem is not true for an infinite interval, — witness the function

$$y = x.$$

And it is not true for a finite interval which is not closed; witness the function

$$y = \frac{1}{x}, \qquad 0 < x \leqq 1.$$

THEOREM 2. *If a function $f(x)$, continuous in a given interval, changes sign, it vanishes at a point of the interval.*

In this theorem it is not necessary that the interval be closed. It is enough that it be connected; i.e. if $a$ and $b$ are two of its points, then the closed interval $(a, b)$ shall belong to the given interval.

Let $a$ be a point of the interval. If $f(a) = 0$, we are through. If not, $f(x)$ must change sign to the right of $a$ or to the left of $a$. Suppose the former is the case. Then there will be an interval,

$$a \leqq x < \gamma$$

in which $f(x)$ does not change sign; Theorem 1, § 8. Let $\xi$ be the upper limit of all such numbers, $\xi$. Then $\xi$ is an interior point of the interval of definition, and

$$f(\xi) = 0.$$

For, if $f(\xi) \neq 0$, then there will be a certain neighborhood of $\xi$ in which $f(x)$ will not change sign. But this is in contradiction with the assumption that $\xi$ is the upper limit.

THEOREM 3. *A function $f(x)$, continuous in a closed interval,*

$$a \leqq x \leqq b,$$

*has a maximum and a minimum.*

By a *maximum* is here meant a value, $M$, taken on by the function in one or more points of the interval, and not exceeded in any point:

$$\begin{array}{lll} f(\xi) = M & \text{for some } \xi, & a \leqq \xi \leqq b; \\ f(x) \leqq M & \text{for every } x, & a \leqq x \leqq b. \end{array}$$

Similarly for a minimum.

For example, the function

$$y = \text{const.}$$

has a maximum in every point $x = \xi$ of the interval; and it also has a minimum in every point.

Again, the function

$$\begin{array}{ll} f(x) = x, & 0 \leqq x \leqq 1; \\ f(x) = 1, & 1 < x \leqq 2; \\ f(x) = 3 - x, & 2 < x \leqq 3 \end{array}$$

has a maximum in each of the points $1 \leqq \xi \leqq 2$ and a minimum in $x = 0, 3$.

*Proof.* Consider the point set $\{y\}$, where

$$y = f(x).$$

It may consist of a single point. In any case, by Theorem 1, it is bounded. Let $M$ denote its upper limit.

The theorem asserts that the point set has a maximum, i. e. that $M$ is a point of the set.

Assume the theorem false. Then $f(a) < M$. Let $\gamma$ be a number between $f(a)$ and $M$:

$$f(a) < \gamma < M.$$

The equation

$$f(x) - \gamma = 0$$

has roots, by Theorem 2, and these form a closed set, by § 8, Ex. 6. Let $\xi$ be the smallest root.

Assign to $\gamma$ a succession of values:

$$\gamma_1 < \gamma_2 < \cdots, \qquad \lim_{n \infty} \gamma_n = M.$$

The corresponding roots will have the property:

$$\xi_1 < \xi_2 < \cdots, \qquad \xi_n < b.$$

Hence $\xi_n$ approaches a limit:

$$\lim_{n=\infty} \xi_n = \xi,$$

and $\xi$ is a point of the interval $(a, b)$, since this is closed. It follows, then, from

$$y_n = f(\xi_n)$$

and the continuity of $f(x)$, with the aid of § 5, Theorem, that

$$\lim y_n = \lim f(\xi_n) = f(\lim \xi_n)$$

or

$$M = f(\xi), \qquad q.e.d.$$

## Exercises

1. Prove Theorem 1 by the method used for Theorem 2; i.e. begin with an interval $a \leqq x \leqq \xi$, in which $f(x)$ is bounded.

2. Prove Theorem 2 by the method used for Theorem 1.

3. Give two new proofs of Theorem 3.

4. Devise a new proof for Theorem 1.

5. The same for Theorem 2.

6. Prove the following theorem. Let $f(x)$ be positive in every point of a closed interval, and let it have the lower limit 0. Show that there is a point of the interval, in every neighborhood of which the lower limit is 0.

7. Let $f(x)$ be defined in the interval

$$a \leqq x \leqq b,$$

and let it be bounded from above. Let it have no maximum. Show that a set of values $x_1, x_2, \cdots$ exists having $c$ as their sole cluster point and such that

$$\lim_{x_n = c} f(x_n) = G,$$

where $G$ is the upper limit of the function $f(x)$.

8. If the function

$$y = f(x)$$

is continuous in the interval

$$a \leqq x \leqq b$$

and is monotonic increasing:

$$f(x') < f(x''), \qquad x' < x'',$$

show that the inverse function:

$$x = F(y),$$

is monotonic increasing and continuous.

**§ 10. Uniform Continuity.** To say that a function, $f(x)$, is continuous at a point $x'$ means that to a positive $\epsilon$ chosen at pleasure corresponds a $\delta > 0$ such that

$$|f(x') - f(x)| < \epsilon, \qquad |x - x'| < \delta, \qquad x, x' \text{ in } A.$$

Let $\epsilon$, once chosen, be held fast, and consider the $\delta$'s corresponding to the different points $x'$ of $A$. These $\delta$'s will in general be different for different $x'$'s, and even when the maximum is chosen each time, it may sink below an arbitrarily chosen positive constant, $h$. For example, if

$$f(x) = \frac{1}{x}, \qquad 0 < x \leqq 1,$$

the value of $\delta$ obviously drops below any given $h$ for some values of $x'$.

But this is not the case, for example, with the function

$$f(x) = x^2, \qquad 0 \leqq x \leqq 1.$$

The worst points $x'$; i. e. those for which $\delta$ has its smallest value, are obviously those for which the graph of the function is steepest — here, $x' = 1$. And so we can set

$$\delta = \frac{1}{2}\epsilon,$$

and this $\delta$ will fit any $x'$ of the interval.

We are thus led to the following

*Definition.* A function $f(x)$ is said to be *uniformly continuous* throughout its domain of definition, $A$, if to an arbitrary positive $\epsilon$ there corresponds a *fixed* positive $\delta$ which will apply to any point $x'$ of $A$:

$$1) \qquad |f(x) - f(x')| < \epsilon, \qquad |x - x'| < \delta; \qquad x, x' \text{ in } A;$$
$$\delta, \text{ independent of } x, x'.$$

THEOREM. *A function $f(x)$ which is continuous at every point of a closed interval $(a, b)$:*

$$a \leqq x \leqq b,$$

*is uniformly continuous throughout that interval.*

*Proof.* Let $\epsilon$ be an arbitrary positive number, once chosen and then held fast. We wish to show that a positive constant, $\delta$, exists such that

$$2)\qquad |f(x)-f(x')| < \epsilon, \qquad |x-x'| < \delta, \qquad x, x' \text{ in } A.$$

When a subinterval

$$A': \qquad a' \leqq x \leqq b', \qquad a \leqq a' < b' \leqq b.$$

is such that, for some fixed $\delta' > 0$, where $\delta'$ depends on $A'$, but not on $x$, $x'$, the relations hold:

$$|f(x)-f(x')| < \epsilon, \qquad |x-x'| < \delta', \qquad x, x' \text{ in } A',$$

we shall say that $A'$ is of *Class* ($\epsilon$). This is the particular $\epsilon$ chosen at the outset. It does not change in the reasoning that follows.

If two intervals, $A'$ and $A''$, each of Class ($\epsilon$), overlap, it is obvious that the composite interval made up of $A'$ and $A''$ is also of Class ($\epsilon$).

There exist intervals of Class ($\epsilon$). For, since $f(x)$ is continuous at $x=a$, we can choose $\epsilon' = \frac{1}{2}\epsilon$ and then find a $\delta_0 > 0$ such that

$$|f(x)-f(a)| < \epsilon', \qquad 0 \leqq x-a < \delta_0.$$

Then

$$|f(x')-f(a)| < \epsilon', \qquad 0 \leqq x'-a < \delta_0.$$

Hence

$$|f(x)-f(x')| < 2\epsilon' = \epsilon,$$

$$0 \leqq x-a < \delta_0, \qquad 0 \leqq x'-a < \delta_0.$$

If, then, we choose $\gamma$ so that $a < \gamma < a+\delta_0$, the interval

$$3)\qquad a \leqq x \leqq \gamma$$

will be of Class ($\epsilon$).

Consider the totality of intervals 3), $\gamma$ now being unrestricted, which are of Class ($\epsilon$). Let $\xi$ be the upper limit of the $\gamma$'s. We will show that $\xi = b$, and that $b$ is a $\gamma$.

Since $f(x)$ is continuous at $\xi$, we can choose $\epsilon' = \frac{1}{2}\epsilon$ and then find a $\delta_1 > 0$ such that

$$|f(x)-f(\xi)| < \epsilon', \qquad |x-\xi| < \delta_1, \qquad x \text{ in } A.$$

We now infer as above that

$$|f(x)-f(x')| < \epsilon,$$

$$|x-\xi| < \delta_1, \qquad |x'-\xi| < \delta_1, \qquad x, x' \text{ in } A.$$

Thus if $\delta_1'$ be chosen between 0 and $\delta_1$, or: $0 < \delta_1' < \delta_1$, the interval

4) $$\xi - \delta_1' \leqq x \leqq \xi + \delta_1', \qquad x \text{ in } A,$$

will be of Class ($\epsilon$).

On the other hand, there is a value $\gamma_1$ of $\gamma$ arbitrarily near $\xi$, and so, in particular,

$$\xi - \delta_1' < \gamma_1.$$

Thus the interval 3) for $\gamma = \gamma_1$, overlaps 4). Hence $\xi = b$ and $b$ is a $\gamma$. This proves the theorem.

*Second Proof.* The theorem can also be proved as follows. Assign to each point $x$ of the interval the maximum value of $\delta$ for which

$$|f(x) - f(x')| < \epsilon, \qquad |x - x'| < \delta, \qquad x' \text{ in } A.$$

Here, $\epsilon$ is the fixed value chosen at the start. Thus a positive function, $\delta = \delta(x)$, is defined in each point of the interval. We wish to show that its lower limit is positive. Suppose it were 0. Then the point set $(x, \delta)$ in the $(x, y)$-plane would have a cluster point $(c, 0)$. In fact, if $(a, 0)$ is not such a point, let the interval $a \leqq x \leqq \gamma \leqq b$ be free from cluster points. The upper limit of $\gamma$ is a point $c$, corresponding to a cluster point.

This leads to a contradiction. For, the function being continuous in $c$, we have:

$$|f(c) - f(x)| < \epsilon', \qquad |x - c| < \delta', \qquad x \text{ in } A;$$

also:

$$|f(c) - f(x')| < \epsilon', \qquad |x' - c| < \delta', \qquad x' \text{ in } A.$$

Hence

$$|f(x) - f(x')| < 2\,\epsilon',$$

where $x$ is any point of the interval

$$c - \frac{1}{2}\delta' < x < c + \frac{1}{2}\delta',$$

and $|x - x'| < \frac{1}{2}\delta'$, both points, $x$ and $x'$, lying in $A$. Now choose $\epsilon' = \frac{1}{2}\epsilon$. Then $\delta(x) \geqq \frac{1}{2}\delta'$ throughout this interval, and so $(c, 0)$ cannot be a cluster point of the set.

*Third Proof.* Still another proof can be given by means of the Covering Theorem, § 11. Begin as in the Second Proof with the system of maximum $\delta$'s. Then the interval can be covered by a finite number of overlapping intervals:

$$\xi - \delta(\xi) < x < \xi + \delta(\xi),$$

where $\xi$ takes on successively each of a finite number of values. If $h$ is the length of the shortest interval common to two consecutive overlapping intervals,

$$\frac{1}{2} h \leqq \delta(x)$$

for every $x$ of the total interval.

## EXERCISES*

1. Show that the function

$$\frac{x^2}{1 + x^2}$$

is uniformly continuous in its domain of definition.

2. Is the function

$$f(x) = e^x$$

uniformly continuous i) in its domain of definition? ii) for positive values of $x$? iii) for negative values of $x$?

3. Show that the function

$$y = \log x$$

is uniformly continuous in the region $1 \leqq x < +\infty$.

4. Is the function $\sqrt{x}$ uniformly continuous?

5. If a function is uniformly continuous, is it bounded?

6. If a continuous function is bounded, is it uniformly continuous?

7. If $f(x)$ is uniformly continuous in the interval

$$a \leqq x < b,$$

show that it approaches a limit when $x$ approaches $b$.

8. If $f(x)$ is continuous in each point of the interval

$$a < x \leqq b$$

and if $f(x)$ approaches a limit when $x$ approaches $a$, show that $f(x)$ is uniformly continuous in the above interval.

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* In working these Exercises make use of what you know about the Calculus.

9. Show that a continuous periodic function is uniformly continuous.

**§ 11. The Covering Theorem.** *To each point $x$ of a closed point set $\{x\}$ let an interval $r_x$ including this point be assigned. Then it is possible to include all points of $\{x\}$ in a finite number of these intervals $r_x$.*

Let $a$ and $b$ be respectively the minimum and the maximum points of $\{x\}$. Let $(a, \gamma)$:

$$a \leqq x \leqq \gamma,$$

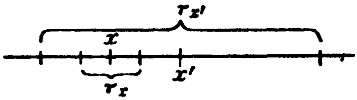
be an interval which can be covered by a finite number of intervals $r_x$. Such intervals exist, for any interval $(a, \gamma)$ included in $r_a$ is one. Let $\xi$ be the upper limit of the $\gamma$'s. Then $\xi = b$ and $b$ is a $\gamma$.

For, since the interval $r_\xi$ extends beyond $\xi$, it is impossible for $\xi$ to be $< b$. Hence $\xi = b$. Consider the interval $r_b$. It extends to the left of $b$, and so $\xi$ cannot be merely an upper limit for the $\gamma$'s — it must itself be a $\gamma$. This completes the proof.

*Second Proof.* A second proof, more convenient in space of $n$ dimensions, is the following. Let $x$ be any point of $\{x\}$. Then a certain $\delta$-neighborhood

$$\sigma: \qquad x_k - \delta < \xi_k < x_k + \delta, \qquad k = 1, \cdots, n,$$

is contained in $r_x$. But a larger neighborhood, not even contained in $r_x$, may be included in some other $r_{x'}$. Let $\delta_x$ be the upper limit for the $\delta$'s corresponding to the point $(x)$; denote the corresponding $\sigma$ by $\sigma_x$.



It is obviously sufficient to show that $\{x\}$ can be covered by a finite number of $\sigma_x$'s. This will surely be so if the lower limit of $\delta_x$ is positive. If this were not true, there would be a point $(\xi)$ of $\{x\}$, in every neighborhood of which $\delta_x$ sank below an arbitrary preassigned $\epsilon$. But this is impossible, since such a neighborhood would come to lie within $\sigma_\xi$.

**§ 12. The Axiom of Choice.** Let $\{x\}$ be a bounded infinite linear point set. We have proved that it must have a point of condensation, $x = c$ (Weierstrass-Bolzano Theorem, § 1). It would seem, then, clear that we can pick out a subset, $x_1, x_2, \cdots$, from its points such that

$$\lim_{n=\infty} x_n = c.$$

For, let $x_1$ be any point of the set distinct from $c$. Choose $\epsilon < |c - x|$ and mark off an interval

$$c - \epsilon < x < c + \epsilon.$$

Choose a point $x_2$ of the set lying in this interval and distinct from $c$. Then choose a smaller interval and repeat the process. Thus a succession of points $x_1, x_2, x_3, \cdots$ is determined.

Now comes the difficulty. In the process just described we have assumed an infinite number of independent choices. What we need is a *law* whereby the $n$-th point is determined *before $n$ is named.* We do not know, for example, what the $n$-th digit in the decimal expression for $\sqrt{2}$ is; but each digit is determined before we say what $n$ shall be.

There seems to be no means of laying down such a law in the general case of the point set just considered. And yet, the existence of such a set of $x_n$ seems highly plausible — it would be little short of perverse to deny its existence. Moreover, an important part of modern analysis has been built up on the tacit assumption that such a set exists.

Zermelo was the first to point out this lacuna. He met the difficulty by introducing a new axiom, which may be formulated as follows.[*]

THE AXIOM OF CHOICE. *Let each point set $A$ in space of $n$ dimensions ~~determine a~~ point $P$ of that space. Then there exists such a determination whereby $P$ is a point of $A$.*

The word *determine* is here used in the same sense as in the case of defining a function. Each point set $A$ is transformed into a point $P$ by a law such that, when $A$ is named subsequently, the law already has its answer, $P$. Thus, to each $A$ might correspond the origin of coordinates. And now the axiom says that, in this manifold of laws, there is one whereby $P$ is a point of $A$ for each $A$.

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[*] Cf. Carathéodory's presentation of this subject, *Reelle Funktionen*, p. 33.

Thus the determination of $P$ in $A$ does not come *after* $A$ has been named, but assigns $P$ to $A$ *before* $A$ enters the specific consideration in hand. Or, still in other words, whenever we select a point set $A$, one of its points, $P$, has already been determined, and is waiting for us.

Returning now to the example with which we opened the discussion, let us proceed more systematically, beginning with a set of positive decreasing $\epsilon$'s:

$$0 < \epsilon_{n+1} < \epsilon_n, \qquad \lim \epsilon_n = 0,$$

and defining intervals

$$R_n: \qquad c - \epsilon_n < x < c + \epsilon_n.$$

Let $A_n$ be the subset of the given set which lies in $R_n$, the point $c$ excepted:

$$0 < |x - c| < \epsilon_n, \qquad x \text{ in } A_n.$$

And now, by the Axiom, there *already exists* a point $x_n$ of $A_n$; it does not have to be chosen after we arrive at $A_n$. It is already there to meet us.

These $x_n$'s define a function of $n$ having the property that

$$|c - x_n| < \epsilon_m, \qquad m \leqq n,$$

and so $x_n$ approaches $c$ as its limit:*

$$\lim_{n=\infty} x_n = c.$$

A further illustration of the use of the Axiom appears in the proof of the following

THEOREM. *An arbitrary infinite point set $M$ in space of $n$ dimensions contains an infinite denumerable set $P_1, P_2, \cdots$.*

*Proof.* Begin with $M$. To it is assigned a point $P$ by the Axiom. Let this be the point $P_1$.

Let $A_1$ be the point set consisting of $M$ less the point $P_1$. Then there is already waiting for us, by the Axiom, a point $P$ of $A_1$. Let this be the point $P_2$.

Repeat the process, taking as $A_2$ the point set $A_1$ less $P_2$. Then a point $P$ of $A_2$ is already determined by the Axiom This shall be the point $P_3$. And so on. We see that the point $P_n$ existed before we began, just as the $n$-th digit in the decimal for $\sqrt{2}$ existed before we began.

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* In this proof, it may happen that the $x_n$'s corresponding to two different values of $n$ are equal. For a given $n$, only a finite number of such coincidences are possible, and they may all be avoided by replacing the total set of intervals $R_n$ by a suitably chosen subset.

## EXERCISES*

1. The *Weierstrass-Bolzano Theorem.* Extend the theorem to point sets in space of $n$ dimensions. Begin with the plane and divide it into squares by the lines

$$x = \frac{p}{2^n}, \quad y = \frac{q}{2^n}.$$

Consider those squares which contain an infinite number of points of the set in their interior and on their boundary. Devise some way of picking out such a square so as to avoid the error of the Auswahlsprinzip (Principle of Choice). Carry the proof through in detail, and generalize.

2. The *$\delta$-Neighborhood of a Point and a Point Set.* Extend the definition given in Chap. II, § 6 and Chap. III, § 5 to a point and a point set $\{x\}$ in space of $n$ dimensions by means of the standard element,

$$a_k - \delta < x_k < a_k + \delta, \qquad k = 1, \cdots, n.$$

3. *Convergence.* Let a sequence of point sets $\{x\}$ in space of $n$ dimensions: $M_1, M_2, \cdots$ be given, their points being denoted by $(x) = (x_1, \cdots, x_n)$. Then $M_k$ shall be said to *converge* when $k = \infty$ if, to an arbitrary positive $\epsilon$ corresponds an integer $m$ and a point $(\xi) = (\xi_1, \cdots, \xi_n)$ of $M_m$ such that each $M_k$ $(m \leqq k)$ lies in the $\epsilon$-neighborhood of $(\xi)$.

Show that there then exists a unique point $(a) = (a_1, \cdots, a_n)$ such that to an arbitrary positive $\epsilon$ corresponds an integer $m$ for which $M_k$ lies in the $\epsilon$-neighborhood of $(a)$ when $m \leqq k$.

The point $(a)$ is called the *limit* of $\{x\}$ or $M_k$:

$$\lim \{x\} = (a) \qquad \text{or} \qquad \lim_{k=\infty} M_k = (a).$$

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[*] These Exercises need not all be worked at this stage. It is well for the student now to extend his horizon. He may well leave the detailed study of this subject until he has become familiar with the application of the theory he has already studied.

4. *Limit of a Function.* Let $f(x_1, \cdots, x_n)$ be defined in the points of $M_k$, Question 3, and let

$$y = f(x_1, \cdots, x_n).$$

Let $\{y\}$ be the succession of point sets $N_k$ corresponding to $M_k$. If $N_k$ converges, the function $f(x_1, \cdots, x_n)$ is said to *converge*, and its limit is defined as $\lim N_k = b$:

$$\lim_{(x)=(a)} f(x_1, \cdots, x_n) = b.$$

By this last equation is meant: i) that $f(x_1, \cdots, x_n)$ converges (*existence* of a limit); ii) that the value of the limit is $b$.

5. Extend the definition of Question 4 to the case that $M_k$ lies in a plane and

$$\lim_{i=\infty} x_i = a, \qquad \lim_{j=\infty} y_j = \infty.$$

Generalize for the two-dimensional case.

6. Give a generalization of Question 5 for higher spaces.

7. When is a function $f(x_1, \cdots, x_n)$ said to be *bounded*, i) in a point; ii) in a region? When does the function have a *pole* in a point?

8. Define *continuity* for a function $f(x_1, \cdots, x_n)$.

9. In what points are the following functions continuous?

$$\text{i)}\ \frac{x-y}{x+y}; \qquad \text{ii)}\ \frac{xy}{x^2+y^2}; \qquad \text{iii)}\ \frac{x^3+y^3}{x^2+y^2}.$$

10. What of the functions in Question 9 approach limits when $(x, y)$ approaches $(0, 0)$?

11. Show that the function

$$\frac{xy}{x^2+y^2}$$

approaches a limit along each straight line through the origin.

12. Extend Theorems 1 and 2 of § 8 to functions of several variables.

13. Show that a rational function:

$$R(x_1, \cdots, x_n) = \frac{f(x_1, \cdots, x_n)}{\varphi(x_1, \cdots, x_n)},$$

where $f(x_1, \cdots, x_n)$ and $\varphi(x_1, \cdots, x_n)$ are polynomials prime to each other, is continuous at all points at which it is defined.

14. Does $R(x_1, \cdots, x_n)$, Question 13, have a pole at a point in which $\varphi(x_1, \cdots, x_n)$ vanishes?

15. Extend Exs. 1 and 6, § 8, to functions of several variables.

16. Extend Theorem 1, § 9, to functions of several variables.

17. The same for Theorem 3, § 9.

18. Theorem 2, § 9, can be generalized as follows. Let $f(x_1, \cdots, x_n)$ be continuous in a connected region of $n$-dimensional space. Let it be positive at one interior point of the region, and negative at another. Then the function has a root in the region.

19. Generalize Exercise 6, § 9.

20. Generalize Exercise 7, § 9.

21. *Uniform Continuity.* A function $f(x_1, \cdots, x_n)$, defined in the points of a point set $A$, is said to the *uniformly continuous* there if to an arbitrary $\epsilon > 0$ corresponds a $\delta > 0$ *independent of* $(x)$ and such that

$$|f(x_1, \cdots, x_n) - f(x_1', \cdots, x_n')| < \epsilon,$$

where $(x')$ is any point of $A$ and $(x)$ is any point of $A$ within the $\delta$-neighborhood of $(x')$.

State and prove the generalization of the Theorem of § 10 for this case.

22. If a function $f(x_1, \cdots, x_n)$ is uniformly continuous in an open region, $R$, and if $P$ is a boundary point of $R$, show that the function approaches a limit when $(x)$ approaches $P$, always remaining in $R$. Assume only such boundaries as are analogous to the surface of a polyhedron in space, or a simple regular curve in the plane, Chap. VII, § 10. Begin with the case $n = 2$.

23. If the definition of the function in Question 22 be supplemented by setting it equal in each boundary point to the limit which it approaches there, show that the extended function is continuous in the closed region.

24. If $f(x_1, \cdots, x_n)$ is defined in the points of a closed set, $A$, and is positive, show that there is a point of $A$, in every neighborhood of which the function comes arbitrarily near its lower limit.

25. *Covering Theorem.* Extend the Covering Theorem to a closed set in space of $n$ dimensions, and prove it.

# Chapter IV

## Derivatives. Integrals. Implicit Functions

**§1. Derivatives.** Let a function $f(x)$ be defined in the neighborhood of a point, $x = x_0$. Form the difference-quotient:

$$1) \qquad \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},$$

where $x_0 + \Delta x$ is a point of the above neighborhood, distinct from $x_0$. If the quotient approaches a limit as $\Delta x$ approaches 0, the function is said to have a *derivative*, or be *differentiable*, at the point $x_0$. We write:

$$2) \qquad \lim_{\Delta x = 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = D_x y = f'(x_0).$$

If the ratio 1) approaches a limit when $\Delta x$ approaches 0 passing only through positive values, $f(x)$ is said to have a *forward derivative.* And similarly for a *backward derivative.* If, and only if, these two are equal, will $f(x)$ have a derivative in the point $x_0$. But if $x_0$ is an end point of the domain of definition of $f(x)$, then $f(x)$ is said to have a derivative in the point $x_0$ if the forward or backward derivative exists.

If a function has a derivative in a point, the function is continuous in the point. But the converse is not true, as will presently be shown.

If the difference-quotient 1) becomes infinite as $\Delta x$ approaches 0, the function is said to have an *infinite derivative.* In particular, we may have

$$\lim_{\Delta x = 0+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = +\infty, \qquad \text{or} \qquad -\infty;$$

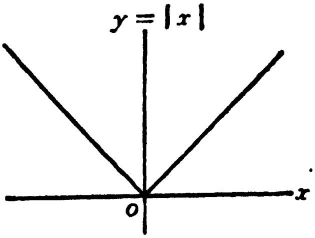
and similarly for $\lim \Delta x = 0^-$. When, however, we say of a function that it *has a derivative*, we shall use the word only in the sense of a proper derivative, and exclude the case that the difference-quotient becomes infinite.

If $f(x)$ has a derivative at every point of an interval, open or not, the function is said to be *differentiable in the interval.*

*Example* 1. The function

$$y = |x|, \quad -\infty < x < \infty,$$

is continuous for all values of $x$. It has a derivative when $x \neq 0$. In the point $x = 0$ it has a forward derivative, the value of which is $+1$; and a backward derivative with the value $-1$. Since $+1 \neq -1$, the function has no derivative at the origin.



*Example* 2. If, however, we take as the domain of definition the interval

$$0 \leqq x < +\infty,$$

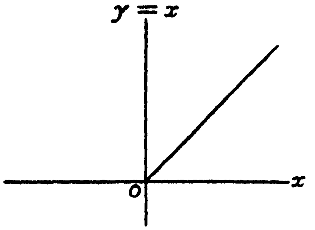
or, again, the interval

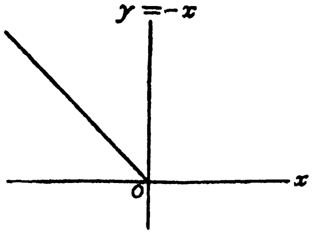
$$-\infty < x \leqq 0,$$

the function

$$y = |x|$$

has, in each case, a derivative at every point.





## EXERCISES

1. If a function has a forward derivative at a point, and also a backward derivative at the point, show that it is continuous at the point.

2. Show that the function[*]

$$f(x) = \frac{x}{2 - e^{1/x}}, \qquad x \neq 0;$$

$$f(0) = 0$$

---

[*] Again we point out that we are using the elementary functions only for the convenience of illustration. The examples could be constructed without them. The theory is in no wise dependent on them. The elementary functions will presently be developed systematically.

is continuous for all values of $x$, but that, at the point $x = 0$, it has no derivative.

3. If each of two functions, $f(x)$ and $\varphi(x)$, has a derivative at the point $x = a$, show that each of the functions

$$f(x) + \varphi(x), \qquad f(x)\,\varphi(x)$$

has a derivative there, also. And the same is true of the function

$$\frac{f(x)}{\varphi(x)},$$

provided $\varphi(a) \neq 0$.

Give an explicit reference to each theorem you use in the proof.

4. Show that the function

$$y = x$$

has a derivative for all values of $x$.

5. The same for

$$y = c,$$

where $c$ is a constant.

6. Prove:

$$D_x x^n = n x^{n-1},$$

where $n$ is a natural number.

7. Show that a polynomial:

$$G(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n,$$

has a derivative.

8. Show that a rational function:

$$R(x) = \frac{G(x)}{F(x)},$$

where $G(x)$, $F(x)$ are polynomials, in general has a derivative. What are the exceptions?

Are $G(x)$, $F(x)$ *any* two polynomials? Answer explicitly.

9. If the function

$$x f(x)$$

has a derivative at a given point, $x_0 \neq 0$, and if $f(x)$ is continuous there, show that $f(x)$ has a derivative there.

10. In Questions 3/9 an $\epsilon$-proof was required in *three* cases. What were they?

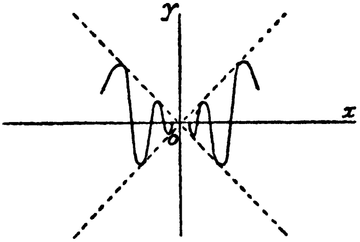
**§2. Continuous Function without a Derivative.** Consider the function

$$1)\qquad \begin{cases} f(x) = x \sin \dfrac{1}{x}, & x \neq 0; \\ f(0) = 0. \end{cases}$$

It is continuous at the origin, but it has no derivative there. For,

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \sin \frac{1}{\Delta x}$$

and this variable approaches no limit, but oscillates between + 1 and − 1 as $\Delta x$ approaches 0.

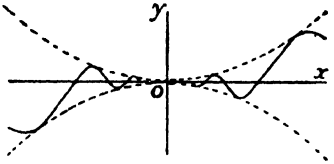


On the other hand, the function

$$2)\qquad \begin{cases} f(x) = x^2 \sin \dfrac{1}{x}, & x \neq 0; \\ f(0) = 0 \end{cases}$$

has a derivative at the origin, since

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \Delta x \sin \frac{1}{\Delta x},$$



and this variable approaches the limit 0.

Observe, however, that

$$\tan \tau = D_x y = f'(x)$$

is not a continuous function. For,

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \qquad x \neq 0,$$

and when $x$ approaches 0, this function approaches no limit. The explanation is simple, when one looks at the graph. For, as a point $P$ moves along the curve, approaching the origin, the tangent oscillates and approaches no limit — why should it? Nevertheless, the curve has a tangent at the origin.

The function

$$3)\qquad \begin{cases} f(x) = x^{\frac{4}{3}} \sin \dfrac{1}{x}, & x \neq 0; \\ f(0) = 0 \end{cases}$$

has a derivative for all values of $x$, but the derivative does not remain finite at the origin. Its value at the origin is 0.

It is easy to form an example of a continuous function which fails to have a derivative in the points of a set everywhere dense; Weierstrass* was the first to give an example of a continuous function which nowhere has a derivative. An example simpler to follow in detail has recently been given by Perkins.**

### § 3. Rolle's Theorem.

*Let* $\varphi(x)$ *be continuous in the closed interval*

$$a \leqq x \leqq b,$$

*and let*

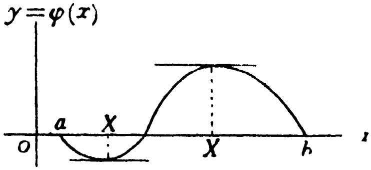
$$\varphi(a) = 0, \qquad \varphi(b) = 0.$$

*Let* $\varphi(x)$ *have a derivative at the interior points,*

$$a < x < b.$$

*Then the derivative vanishes at an interior point:*

$$\varphi'(X) = 0, \qquad a < X < b.$$



*Proof.* If $\varphi(x) \equiv 0$, the truth of the theorem is obvious. In all other cases the function will take on positive or negative values, or both. Hence the function will have a maximum or a minimum inside the interval; Chap. III, § 9, Theorem 3. Suppose it has a maximum at $x = X$. Then

$$\varphi(X + h) \leqq \varphi(X)$$

for all values of $h$ numerically small. Thus

$$\frac{\varphi(X+h) - \varphi(X)}{h} \begin{cases} \leqq 0, & 0 < h; \\ \geqq 0, & h < 0. \end{cases}$$

---

* Cf. C. Wiener, *Journ. für Math.* vol. 90 (1881) p. 221.

** F. W. Perkins, *Amer. Math. Monthly* vol. 34 (1927) p. 476.

Since $\varphi(x)$ by hypothesis has a derivative at $x = X$, this difference-quotient must approach a limit as $h$ approaches 0. The forward derivative is $\leqq 0$; the backward derivative is $\geqq 0$. Hence the derivative is 0, q. e. d.

The importance of this theorem lies in the fact that its proof is purely *arithmetic*, not based on geometric intuition, but solely on the theory here developed analytically.

**§ 4. Law of the Mean.** *Let $f(x)$ be a function continuous in the closed interval $a \leqq x \leqq b$, and let it have a derivative at each interior point: $a < x < b$. Then*

$$f(b) - f(a) = (b - a) f'(X), \qquad a < X < b;$$

*or*

$$f(a + h) = f'(a) + h f'(a + \theta h), \qquad 0 < \theta < 1.$$

*Proof.* Form the function:

$$\varphi(x) = (x - a)\,[f(b) - f(a)] - (b - a)\,[f(x) - f(a)].$$

This function satisfies all the conditions of Rolle's Theorem. Hence its derivative,

$$\varphi'(x) = f(b) - f(a) - (b - a)\,f'(x),$$

must vanish within the interval:

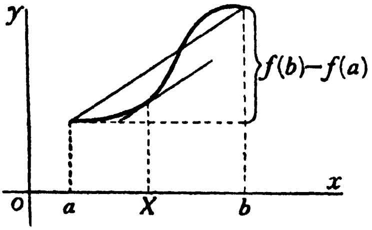
$$\varphi'(X) = f(b) - f(a) - (b - a) f'(X) = 0, \qquad a < X < b.$$

From this equation the theorem follows at once.

Again, a theorem the truth of which is, geometrically, intuitively obvious — for must not a tangent be parallel to the secant, or

$$\frac{f(b) - f(a)}{b - a} = f'(X)\,?$$

— has been proved *arithmetically* with the analytic means here at our disposal.



As a first application consider the theorem that *a function $f(x)$ whose derivative vanishes identically, is a constant.* Let $x$ and $a$ be two points such that the closed interval $(a, x)$ lies in the domain of definition. Then

$$f(x) - f(a) = (x - a)\,f'(X) = 0.$$

A corollary of this theorem is the fact that *if two functions, $f(x)$ and $\varphi(x)$, have the same derivative,*

$$f'(x) = \varphi'(x),$$

*they differ by a constant:*

$$f(x) = \varphi(x) + c.$$

## § 5. Differentiation of Composite Functions. Differentials.

Let

$$u = f(y)$$

be a function of $y$, defined in the neighborhood of the point $y = y_0$ and having a derivative in that neighborhood. Let

$$y = \varphi(x)$$

be a function of $x$, defined in the neighborhood of the point $x = x_0$ and having a derivative in that point. Finally, let

$$\varphi(x_0) = y_0.$$

Then $u$, regarded as a function of $x$:

$$u = f[\varphi(x)],$$

has a derivative in the point $x = x_0$, and

$$D_x u = D_y u \ D_x y.$$

For, let

$$\Delta y = \varphi(x_0 + \Delta x) - \varphi(x_0),$$

where $|\Delta x| < h$ and $h$ is so chosen i) that the points

$$|x - x_0| < h$$

lie in the second neighborhood and ii) that the point $y_0 + \Delta y$ lies in the first neighborhood. Then, by the Law of the Mean,

$$\Delta u = f(y_0 + \Delta y) - f(y_0) = f'(y_0 + \theta \Delta y) \Delta y.$$

Hence

$$\frac{\Delta u}{\Delta x} = f'(y_0 + \theta \Delta y) \frac{\Delta y}{\Delta x}.$$

When $\Delta x$ approaches 0, the right hand side approaches a limit, provided that $f'(y)$ is continuous:

$$\lim_{\Delta x = 0} f'(y_0 + \theta \Delta y) \frac{\Delta y}{\Delta x} = \lim_{\Delta y = 0} f'(y_0 + \theta \Delta y) \lim_{\Delta x = 0} \frac{\Delta y}{\Delta x}$$

$$= f'(y_0)\, \varphi'(x_0).$$

This proves the theorem under the restriction mentioned.

The proof given in the elementary treatment of the Calculus by writing

$$\frac{\Delta u}{\Delta x}=\frac{\Delta u}{\Delta y}\,\frac{\Delta y}{\Delta x}$$

is not general, since $\Delta y$ may vanish for values of $\Delta x \neq 0$ in every neighborhood of the point $\Delta x = 0$. For example,

$$\varphi(x)=c,$$

where $c$ is a constant. Here, $\Delta y \equiv 0$, and so for no value of $\Delta x$ can we divide by $\Delta y$.

It is possible to meet the difficulty and establish the theorem without any restriction. Let

$$\psi(\Delta y)=\frac{f(y_0+\Delta y)-f(y_0)}{\Delta y}, \qquad \Delta y \neq 0;$$
$$\psi(0)=f'(y_0).$$

Then the equation:

$$\frac{\Delta u}{\Delta x}=\psi(\Delta y)\,\frac{\Delta y}{\Delta x}$$

is true for all values of $\Delta x$ considered. Now, the function $\psi(y)$ is continuous at the point $y=0$. Hence the right hand side of this equation approaches a limit:

$$\lim_{\Delta x=0}\psi(\Delta y)\frac{\Delta y}{\Delta x}=f'(y_0)\,\varphi'(x_0),$$

and the proof is now complete.

*Differentials.* It is to the theorem just proved that the differentials, regarded as an aid to differentiation, owe their value. For, by definition,

$$dy=D_x y\;\Delta x,$$

when $x$ is the independent variable. Moreover, when $x$ is the independent variable, we define $dx$ as equal to $\Delta x$:

$$dx=\Delta x.$$

Hence

$$dy=D_x y\;dx,$$

when $x$ is the independent variable. And now, by the theorem just proved, it follows that this equation is true when $x$ and $y$ are both functions of $t$:

$$y=f(x), \qquad x=\varphi(t).$$

For, $$D_t y = D_x y \, D_t x,$$
and by definition:

$$dy = D_t y \; \Delta t, \qquad dx = D_t x \; \Delta t.$$

**§ 6. Taylor's Theorem with a Remainder.** *Let* $f(x)$ *be continuous in the closed interval*

$$a \leqq x \leqq b$$

*Let* $f(x)$ *have derivatives of the first* $n + 1$ *orders,*

$$f'(x), \quad f''(x), \quad \cdots, \quad f^{n+1}(x),$$

*at all interior points of the interval:* $a < x < b$. *Then*

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots + \frac{h^n}{n!} f^{n)}(x_0)$$

$$+ \frac{h^{n+1}}{(n+1)!} f^{n+1)}(x_0 + \theta h), \qquad 0 < \theta < 1,$$

*where* $x_0$, $x_0 + h$ *are any two points of the interval.*

The proof is given in the Calculus; cf. for example the Author's *Introduction to the Calculus*, p. 431.

**§ 7. Functions of Several Variables.** Let

$$u = f(x_1, \cdots, x_n)$$

be a function defined in the points of an open region $R$ in the $n$-dimensional space of the variables $(x_1, \cdots, x_n)$. The concepts: *convergence, limit, continuity* have already been defined, Chap. III, end, Exercises 2/8, pp. 94/95. The partial derivatives of $u$ at the points of $R$ are defined in the usual manner; e. g.

$$\frac{\partial u}{\partial x_1} = \lim_{\Delta x_1 = 0} \frac{f(x_1 + \Delta x_1, x_2, \cdots, x_n) - f(x_1, x_2, \cdots, x_n)}{\Delta x_1}.$$

If $$u = f(x, y),$$
we shall write the derivative in either one of the rotations:

$$\frac{\partial u}{\partial x} = f_x(x, y) \qquad \text{or} \qquad f_1(x, y), \qquad \text{etc.}$$

Similarly:

$$\frac{\partial^2 u}{\partial x \partial y} = f_{xy}(x, y) \qquad \text{or} \qquad f_{12}(x, y), \qquad \text{etc.}$$

The proof that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

under suitable restrictions — the existence and continuity of $f_x(x, y)$, $f_y(x, y)$, $f_{xy}(x, y)$ is sufficient — will be given in Chap. IX, § 6.

If there is only one independent variable, $n = 1$, the derivative becomes an ordinary, or total derivative:

$$\frac{du}{dt}.$$

But it would be inconvenient to make an exception in this case, and so we write

$$\frac{\partial u}{\partial t}$$

when occasion arises.

*Law of the Mean.* Let the function $f(x_1, \cdots, x_n)$ be defined in the region

$$\mathcal{R}: \qquad |x_k - a_k| < A_k, \qquad k = 1, \cdots, n,$$

and let it have partial derivatives of the first order there. Let

$$|h_k| < A_k, \qquad k = 1, \cdots, n.$$

Then the difference

$$f(a_1 + h_1, \cdots, a_n + h_n) - f(a_1, \cdots, a_n)$$

can be written in the form:

$$\begin{aligned} &f(a_1 + h_1, a_2 + h_2, \cdots, a_n + h_n) - f(a_1, a_2 + h_2, \cdots, a_n + h_n) \\ &\quad + f(a_1, a_2 + h_2, \cdots, a_n + h_n) - f(a_1, a_2, a_3 + h_3, \cdots, a_n + h_n) \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\qquad + f(a_1, \cdots, a_{n-1}, a_n + h_n) - f(a_1, \cdots, a_{n-1}, a_n). \end{aligned}$$

Applying the law of the mean for functions of a single variable to each of these differences we arrive at a first form of the Law of the Mean for functions of several variables:

$$1) \qquad \begin{aligned} &f(a_1 + h_1, \cdots, a_n + h_n) - f(a_1, \cdots, a_n) \\ &= \sum_{k=1}^{n} h_k\, f_k(a_1, \cdots, a_{k-1}, a_k + \theta_k h_k, a_{k+1} + h_{k+1}, \cdots, a_n + h_n), \\ &\qquad 0 < \theta_k < 1. \end{aligned}$$

From this result we infer, as a first application, that if the partial derivatives are bounded in the point $(a)$, the function is continuous there. But the theorem would not be true without the restriction, as appears from the example:

$$2) \qquad \begin{cases} f(x,y) = \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0); \\ f(0,0) = 0. \end{cases}$$

With the aid of the Law of the Mean the theorem relating to Change of Variables (Theorem 1) is established, and by means of it in turn a more symmetric form of the Law of the Mean is obtained in case the partial derivatives are continuous in $R$. Let

$$F(t) = f(a_1 + t\,h_1, \cdots, a_n + t\,h_n), \qquad 0 \leqq t \leqq 1.$$

Apply the law of the mean to $F(t)$:

$$F(1) - F(0) = F'(\theta), \qquad 0 < \theta < 1.$$

Hence

$$3) \qquad f(a_1 + h_1, \cdots, a_n + h_n) - f(a_1, \cdots, a_n)$$
$$= \sum_{k=1}^{n} h_k\, f_k(a_1 + \theta\,h_1, \cdots, a_n + \theta\,h_n), \qquad 0 < \theta < 1.$$

In this form the law is easily remembered. It holds for more general regions. Let $S$ be a region which contains in its interior the points of the line

$$x_k = a_k + t\,h_k, \qquad k = 1, \cdots, n, \qquad 0 \leqq t \leqq 1.$$

Then the above proof applies and the theorem is true.

THEOREM 1. CHANGE OF VARIABLES. *Let*

$$u = f(y_1, \cdots, y_n)$$

*be continuous, together with its partial derivatives of the first order, in the neighborhood of the point* $(y) = (b)$; *i. e.* $B$: $(b_1, \cdots, b_n)$. *Let*

$$\varphi_k(x_1, \cdots, x_m), \qquad k = 1, \cdots, n,$$

*be continuous, together with its partial derivatives of the first order, in the neighborhood of the point* $(x) = (a)$; *and let*

$$\varphi_k(a_1, \cdots, a_m) = b_k, \qquad k = 1, \cdots. n.$$

*Let*

$$y_k = \varphi_k(x_1, \cdots, x_m), \qquad k = 1, \cdots, n,$$

*be substituted in the function* $f(y_1, \cdots, y_n)$, *u thus becoming a function of* $(x_1, \cdots, x_m)$. *Then*

$$4) \qquad \frac{\partial u}{\partial x_r} = \frac{\partial u}{\partial y_1}\frac{\partial y_1}{\partial x_r} + \cdots + \frac{\partial u}{\partial y_n}\frac{\partial y_n}{\partial x_r}.$$

It is of fundamental importance to observe, in this last formula, what the *independent variables* are. There are two classes:

Class $A$, the $(x_1, \cdots, x_m)$;

Class $B$, the $(y_1, \cdots, y_n)$.

Those partial derivatives in which an $x_r$ appears below the line, assume a function of the variables of Class $A$; those in which a $y_k$ appears below the line, assume a function of the variables of Class $B$.

The student will do well to illumine this last formula, writing each $x_r$, say, in red ink, and each $y_j$ in blue.

For the proof of the theorem the student is referred to treatises on the Calculus; cf. e. g. the Author's *Advanced, Calculus*, Chap. V.

*Differentials.* Let

$$u = f(x_1, \cdots, x_n)$$

be defined in the above region $R$ and possess partial derivatives of the first order there. Let $(x)$ and $(x + \Delta x) = (x_1 + \Delta x_1, \cdots, x_n + \Delta x_n)$ be any two points of the region. Then

$$\Delta u = f(x_1 + \Delta x_1, \cdots, x_n + \Delta x_n) - f(x_1, \cdots, x_n)$$

can be written by the Law of the Mean, 1), in the form:

$$5) \qquad \Delta u = \bar{f}_1 \Delta x_1 + \cdots + \bar{f}_n \Delta x_n,$$

where $\bar{f}_k$ denotes the value of $f_k$ formed for the mean point of that formula.

If, in particular, the derivatives $f_k(x_1, \cdots, x_n)$ are continuous in $R$, set

$$\bar{f}_k = f_k + \zeta_k, \qquad k = 1, \cdots, n,$$

where $f_k$ is formed for the point $(x)$. Thus

$$6) \qquad \Delta u = \frac{\partial u}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial u}{\partial x_n}\Delta x_n$$

$$+ \zeta_1 \Delta x_1 + \cdots + \zeta_n \Delta x_n.$$

The first line on the right is called the *principal part* of the infinitesimal $\Delta u$ and is defined as the *differential* of $u$:

7) $$du = \frac{\partial u}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial u}{\partial x_n}\Delta x_n.$$

The differential of each of the independent variables $x_k$ is defined as the increment,

$$dx_k = \Delta x_k, \qquad k = 1, \cdots, n.$$

Thus 7) becomes:

8) $$du = \frac{\partial u}{\partial x_1} dx_1 + \cdots + \frac{\partial u}{\partial x_n} dx_n.$$

And now the fundamental theorem about differentials is, that 8) *is true, no matter what the independent variables may be.*

More explicitly, consider the change of variables defined above, the notation being that of Theorem 1. Then

9) $$du = \sum_{\alpha=1}^{n} \frac{\partial u}{\partial y_\alpha} dy_\alpha,$$

the $y_\alpha$ being the independent variables. Secondly,

10) $$dy_\alpha = \sum_{r=1}^{m} \frac{\partial y_\alpha}{\partial x_r} dx_r,$$

the $x_r$ being the independent variables. Now, when the $x_r$ are the independent variables,

11) $$du = \sum_{r=1}^{m} \frac{\partial u}{\partial x_r} dx_r,$$

By Theorem 1,

$$\sum_{r=1}^{m} \frac{\partial u}{\partial x_r} dx_r = \sum_{r=1}^{m} \sum_{\alpha=1}^{n} \frac{\partial u}{\partial y_\alpha} \frac{\partial y_\alpha}{\partial x_r} dx_r.$$

The left-hand side of this equation is equal to $du$ as given by 11). The right-hand side is equal, by virtue of 10), to $du$ as given by 9). Hence 9) is true, regardless of whether the independent variables are those of Class $A$ or Class $B$, and that is what we set out to prove. The result can be stated as

THEOREM 2. DIFFERENTIALS. *Let*

$$u = f(y_1, \cdots, y_n)$$

*and let a change of variables:*

$$y_k = \varphi_k(x_1, \cdots, x_m), \qquad k = 1, \cdots, m,$$

*be made under the conditions stated above. Then*

$$du = \frac{\partial u}{\partial y_1} dy_1 + \cdots + \frac{\partial u}{\partial y_n} dy_n, \tag{12}$$

*regardless of whether the independent variables are those of Class A or Class B.*

*Remark.* Since the differentials of the independent variables by definition are the same as the increments, they are arbitrary. Hence an equation:

$$A_1\, dz_1 + \cdots + A_p\, dz_p = B_1\, dz_1 + \cdots + B_p\, dz_p,$$

where $z_1, \cdots, z_p$ are the independent variables and $A_\alpha$, $B_\alpha$ are any functions of $(z_1, \cdots, z_p)$, leads to the inference that

$$A_\alpha = B_\alpha, \qquad \alpha = 1, \cdots, p.$$

**§ 8. Integral of a Continuous Function.** Let $f(x)$ be a function, continuous in the closed interval

$$a \leqq x \leqq b.$$

Mark the points $x = p/2^n$, where $n$ is a natural number and $p = 0, \pm 1, \pm 2, \cdots$. For a fixed $n$, those points which lie within the interval shall be denoted by:

$$a < x_1 < x_2 \cdots < x_\nu < b.$$

Furthermore, set

$$x_0 \leqq a, \qquad b \leqq x_{\nu+1};$$

$$\Delta x_k = x_{k+1} - x_k, \quad k = 0, 1, \cdots, \nu; \qquad \Delta x_k = 2^{\;n}.$$

Let $x'_k$ be any point of the interval $(a, b)$, which lies in the closed interval $(x_k, x_{k+1})$:

$$x_k \leqq x'_k \leqq x_{k+1}.$$

Form the sum:[*]

$$S = \sum_{k=p}^{q} f(x'_k)\, \Delta x_k,$$

where $p$ is either of the numbers 0, 1; and $q$ is $\nu - 1$ or $\nu$. When $n$ becomes infinite, the sum $S$ approaches a limit, $I$, and $I$ is defined to be the *value* of the *definite integral*

---

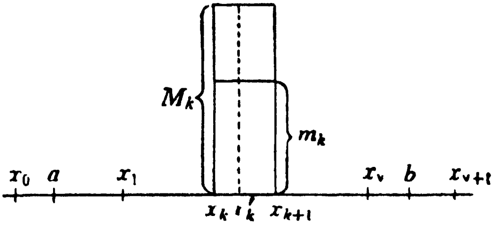
[*] When there is no point of division within the interval, $S$ still is defined if $p = 0$, $q = 1$. For other values of $p$, $q$ let $S = 0$.

$$\int_a^b f(x)\,dx,$$

or

$$\int_a^b f(x)\,dx = \lim_{n=\infty} \sum_{k=p}^{q} f(x'_k)\,\Delta x_k.$$

To prove the convergence we proceed as follows. Let $M_k$, $m_k$ be the maximum and the minimum values of $f(x)$ in the closed interval $(x_k, x_{k+1})$. Assume first that $f(x) > 0$ and let $M$, $m$ be the maximum and the minimum of $f(x)$ in $(a, b)$:



$$m \leqq f(x) \leqq M.$$

Form the sum:

$$T_n = \sum_{k=0}^{\nu} M_k\,\Delta x_k.$$

As $n$ increases, $T_n$ decreases or remains unchanged:

$$T_n \geqq T_{n+1}.$$

But

$$T_n \geqq \sum_{k=0}^{\nu} m\,\Delta x_k \geqq m(b-a).$$

Hence $T_n$ approaches a limit, Chap. II, § 5:

$$\lim_{n=\infty} T_n = I_1.$$

Similarly, the sum

$$t_n = \sum_{k=1}^{\nu-1} m_k\,\Delta x_k$$

increases with $n$, or remains constant:

$$t_n \leqq t_{n+1};$$

and

$$t_n \leqq \sum_{k=1}^{\nu-1} M\,\Delta x_k \leqq M(b-a).$$

Hence $t_n$ approaches a limit:

$$\lim_{n=\infty} t_n = I_2.$$

Moreover,

$$t_n \leqq T_n,$$

and hence

$$I_2 \leqq I_1.$$

Finally,

$$I_2 = I_1.$$

For,

$$T_n - t_n = \sum_{k=1}^{\nu-1} (M_k - m_k)\,\Delta x_k + M_0\,\Delta x_0 + M_\nu\,\Delta x_\nu.$$

Since $f(x)$, being continuous in the closed interval $(a, b)$, is uniformly continuous, Chap. III, § 10, it follows that, to a positive $\epsilon$ chosen at pleasure, there corresponds a positive $\delta$ such that

$$|f(x) - f(x')| < \epsilon, \qquad |x - x'| < \delta, \qquad x, x' \text{ in } (a, b).$$

If, then, $\Delta x_k = 2^{-n}$ is less than $\delta$, the first sum is less than $\epsilon(b - a)$. Hence

$$T_n - t_n < \epsilon(b - a) + 2M\,2^{-n}.$$

Now

$$I_1 \leqq T_n, \qquad t_n \leqq I_2,$$

and so

$$0 \leqq I_1 - I_2 < \epsilon(b - a) + 2M\,2^{-n}.$$

Hence

$$I_1 = I_2.$$

Turning now to the sum $S$ we see that

$$\sum_{k=1}^{\nu-1} m_k\,\Delta x_k \leqq \sum_{k=p}^{q} f(x'_k)\,\Delta x_k \leqq \sum_{k=0}^{\nu} M_k\,\Delta x_k.$$

Since each of the extreme sums approaches the common limit $I_1 = I_2 = I$, it follows that the sum $S$ approaches this same limit.

It remains to remove the restriction $f(x) > 0$. Let $C$ be a constant such that the function

$$\varphi(x) = f(x) + C$$

is positive in $(a, b)$. Form the sums:

$$\sum_{k=p}^{q} \varphi(x'_k)\,\Delta x_k = \sum_{k=p}^{q} f(x'_k)\,\Delta x_k + C(x_q - x_p).$$

The sum on the left converges when $n$ becomes infinite, as has just been shown. The last term on the right approaches a limit. Hence the sum $S$ coverges here, also.

From the method employed in formulating this existence theorem follow at once two corollaries.

COROLLARY 1. LAW OF THE MEAN:

$$\int_a^b f(x)\,dx = (b-a)f(X), \qquad a < X < b.$$

For, obviously:

$$(b-a)\,m \leqq \int_a^b f(x)\,dx \leqq (b-a)\,M,$$

the lower signs holding only when $f(x) = \text{const.} = m = M$. If we set

$$\int_a^b f(x)\,dx = (b-a)\,Y,$$

then

$$m < Y < M,$$

unless $f(x)$ is a constant. Hence the function

$$f(x) - Y,$$

which is continuous in the closed interval $(a, b)$, changes sign there, and so has a root, $x = X$, within the interval, Chap. III, § 9, Theorem 2. — In the excepted case, $X$ may have any value within the interval.

COROLLARY 2. *If* $a < c < b$, *then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The proof is immediate, thanks to the formulation of the existence theorem, and may be left to the student.

We are now in a position to prove the following

CONVERGENCE THEOREM. *Let* $f(x)$ *be continuous in the closed interval* $(a, b)$. *Let the interval be divided in any manner into* $n$ *subintervals by the points*

$$a < x_1 < \cdots < x_{n-1} < b,$$

*and let* $x_0 \leqq a$, $b \leqq x_n$; *let*

$$\Delta x_k = x_{k+1} - x_k, \qquad x_k \leqq x_k' \leqq x_{k+1}, \qquad x_k' \text{ in } (a, b).$$

*Then the sum*

$$\sum_{k=p}^{q} f(x_k') \, \Delta x_k,$$

*where* $p = 0$ *or* $1$ *and* $q = n - 2$ *or* $n - 1$, *approaches a limit for* $n = \infty$, *provided the longest* $\Delta x_k$ *approaches* $0$.

*The value of this limit is the value of the definite integral:*

$$\lim_{n=\infty} \sum_{k=p}^{q} f(x_k') \, \Delta x_k = \int_a^b f(x) \, dx.$$

*Proof.* Consider first the case: $x_0 = a$, $x_n = b$, $p = 0$, $q = n - 1$. The general case then follows at once. By Corollaries 2 and 1,

$$\int_a^b f(x) \, dx = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} f(x) \, dx = \sum_{k=0}^{n-1} f(X_k) \, \Delta x_k,$$

$$x_k < X_k < x_{k+1}.$$

Hence

$$\int_a^b f(x) \, dx - \sum_{k=0}^{n-1} f(x_k') \, \Delta x = \sum_{k=0}^{n-1} [f(X_k) - f(x_k')] \, \Delta x_k.$$

Since $f(x)$ is continuous in the closed interval $(a, b)$, it is uniformly continuous, and so

$$|f(X_k) - f(x_k')| < \epsilon, \qquad |\Delta x_k| < \delta.$$

Thus

$$\left| \int_a^b f(x) \, dx - \sum_{k=0}^{n-1} f(x_k') \, \Delta x_k \right| < \epsilon (b - a),$$

and the theorem is proved.

On the basis of the Convergence Theorem one may define the definite integral as the limit of the more general sum which appears in this theorem. In any case, the definite integral is the *mark*:

$$\int_a^b f(x) \, dx.$$

Its *value* is the limit approached by the sum in question, and this number is represented by the same mark.

*Extension of the Definition.* If $b < a$, the definite integral shall be defined as follows:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

If $a \leqq c \leqq b$, then, by definition:

$$\int_c^c f(x)\,dx = 0.$$

Hitherto we have considered only functions $f(x)$ defined in every point of a closed interval. We now extend the definition to the case that, in a finite number of points, the function is not defined or discontinuous; being continuous, however, in all other points, and bounded in the interval. In particular, the interval may be open at one or at both ends, as

$$a < x < b;$$

but it must be bounded. The above definition of the definite integral, and the Convergence Theorem, apply to such functions, with the obvious modification that no $x'_k$ can be chosen at a point where $f(x)$ is not defined, and that a maximum (minimum) may have to be replaced by an upper (lower) limit.

*Indefinite Integral.* Let $f(x)$ be continuous in the open or closed interval $(a, b)$ and let $F(x)$ be a function whose derivative is equal to $f(x)$:

$$F'(x) = f(x).$$

Then $F(x)$ is called the *indefinite integral* of $f(x)$.

One such function is

$$F(x) = \int_a^x f(t)\,dt.$$

For, by the Law of the Mean .

$$F(x + \Delta x) - F(x) = \int_x^{x+\Delta x} f(t)\,dt = \Delta x\, f(x + \theta\Delta x),$$

$$\lim_{\Delta x = 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x = 0} f(x + \theta\Delta x) = f(x).$$

The most general function is (Chap. IV, § 4):

$$F(x) = \int_a^x f(t)\,dt + C.$$

### EXERCISES

Let the functions $f(x)$, $\varphi(x)$, etc. be defined in the closed interval $a \leqq x \leqq b$, with the possible exception of a finite number of points. Let them be continuous except at most for a finite number of points, at each of which the function may or may not be defined. And let the functions be bounded in the interval.

Prove the following theorems:

1) $$\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx,$$

where $c$ is a constant.

2) $$\int_a^b [f(x) + \varphi(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b \varphi(x)\,dx.$$

3) $$\left| \int_a^b f(x)\,dx \right| \leqq \int_a^b |f(x)|\,dx.$$

4) $$\left| \int_a^b [f(x) + \varphi(x)]\,dx \right| \leqq \int_a^b |f(x)|\,dx + \int_a^b |\varphi(x)|\,dx.$$

5) $$F(x) = \int_a^x f(t)\,dt$$

is continuous, $a \leqq x \leqq b$.

6) $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

provided $f(t)$ is continuous at $t = x$.

7) $$\int_a^b f'(x)\,dx = f(b) - f(a),$$

provided $f'(x)$ is a function belonging to the class here considered, and $f(x)$ is continuous, $a \leqq x \leqq b$.

8) $$\int_a^b f(x)\,\varphi'(x)\,dx = f(x)\,\varphi(x)\Big|_a^b - \int_a^b \varphi(x)\,f'(x)\,dx,$$

provided $f'(x)$, $\varphi'(x)$ belong to the class of functions here considered, and $f(x)\,\varphi(x)$ is continuous, $a \leqq x \leqq b$.

Obtain a generalization of 8) for certain discontinuous functions. Suggestion:

9) $$\int_a^b f(x)\,\varphi'(x)\,dx = \sum_k f(x)\,\varphi(x)\Big|_{\xi_k^+}^{\xi_{k+1}^-} - \int_a^b \varphi(x)\,f'(x)\,dx.$$

**§9. Implicit Functions.** In the Calculus we learn how to differentiate an implicit function. Thus if

1) $$F(x, y, z) = 0,$$

2) $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

But how do we know that Equation 1) has a solution; i.e. defines a function? If, for example,

$$F(u, x) = u^2 + x^2,$$

the equation

$$F(u, x) = 0$$

is true for $x = 0$, $u = 0$; but for no other value of $x$ does it have a root.

Again, if

$$F(u, x) = u^2 - x^2,$$

then the equation

$$F(u, x) = 0$$

is satisfied by the two single-valued functions:

$$u = x, \qquad u = -x;$$

and in the neighborhood of the origin both functions must be retained, to give the complete solution. So, even when the equation

$$F(u, x) = 0$$

has a solution, there is no guarantee that the solution will be given by a single-valued function.

When I was a student, I learned in Williamson's *Differential Calculus* (the best seller of that day) that the envelope of a family of curves,

$$f(x, y, \alpha) = 0,$$

is found by differentiating partially with respect to $\alpha$:

$$\frac{\partial f}{\partial \alpha} = 0,$$

and then eliminating $\alpha$ between the two equations. Thus the family of circles:

$$(x - \alpha)^2 + y^2 = r^2$$

have an envelope obtained by differentiating partially:

$$-2(x - \alpha) = 0,$$

and then eliminating $\alpha$:

$$y^2 = r^2$$

or

$$y = r, \qquad y = -r.$$

But I wondered what would happen if the equation of the family were thrown into the equivalent form:

$$\varphi(x, y) = \alpha;$$

or in the case of the example:

$$x - \alpha = \pm\sqrt{r^2 - y^2}.$$

The rule led to $1 = 0$, and that did not seem quite right; but it did not disturb Williamson, if indeed he had ever thought of it. Youth is iconoclastic, and the Method of Envelopes was one of the Articles of Faith, in those days, to which I could not subscribe.

The reader will have gathered from these remarks that there is real need of an answer to the question: When does the equation

$$F(u, x) = 0$$

or

$$F(u, x_1, \cdots, x_n) = 0$$

define a function $u$ of $x$ or $x_1, \cdots, x_n$? We proceed to the answer.

## § 10. The Existence Theorem.* *Let*

1) $$F(u, x, y, \cdots)$$

*be a single-valued, continuous function in the neighborhood* $(A)$ *of the point* $(u_0, x_0, y_0, \cdots)$:

$$|u - u_0| < A, \quad |x - x_0| < A, \quad |y - y_0| < A, \cdots, \quad 0 < A;$$

*and let*

2) $$F(u_0, x_0, y_0, \cdots) = 0.$$

*Let* $F(u, x, y, \cdots)$ *possess first partial derivatives, and let them be continuous, in the above region* $(A)$.

*Finally, let*

$$\left.\frac{\partial F}{\partial u}\right|_{(0)} = F_u(u_0, x_0, y_0, \cdots) \neq 0.$$

*Then the equation:*

3) $$F(u, x, y, \cdots) = 0$$

*has the following solution in a certain neighborhood of the point* $(u_0, x_0, y_0, \cdots)$:—

*There exists a function,*

4) $$u = \varphi(x, y, \cdots),$$

*single-valued and continuous in a region*

$$|x - x_0| < h, \quad |y - y_0| < h, \cdots, \qquad 0 < h \leqq A,$$

*and having the properties:*

a) $$\varphi(x_0, y_0, \cdots) = u_0, \qquad |\varphi(x, y, \cdots)| < A;$$

b) *When* $\varphi$ *is substituted for* $u$ *in* $F$, *this function vanishes identically:*

5) $$F[\varphi(x, y, \cdots), x, y, \cdots] \equiv 0;$$

c) *The only roots* $(u, x, y, \cdots)$ *of Equation* 3) *which lie in the region*

$$|u - u_0| < A', \quad |x - x_0| < h, \quad |y - y_0| < h, \cdots$$

*where* $0 < A' \leqq A$, *are those for which*

6) $$u = \varphi(x, y, \cdots).$$

---

* The first proof of the theorem, under narrower hypotheses, is due to Cauchy, Turin Memoir of 1831; cf. *Exercices d'analyse* vol. 2 (1841) p. 65. In its present formulation the proof was given by Dini, *Analisi infinitesimale* vol. 1 (1877/78) p. 162.

*Finally, the function* $\varphi(x, y, \cdots)$ *has continuous derivatives of the first order given by the rule of the Calculus.*

*Proof.* We will begin with the case that $(x, y, \cdots)$ reduce to a single variable $x$:

$$7)\qquad F(u, x).$$

Here, we can follow each step by a geometric representation. The region $(A)$ is interpreted by the interior of a square with its vertices in the four points $(u_0 \pm A, x_0 \pm A)$. Let

$$0 < F_u(u_0, x_0).$$

Since $F_u(u, x)$ is continuous, it is possible to find a subregion $(A')$:

$$|u - u_0| \leqq A', \qquad |x - x_0| \leqq A', \qquad 0 < A' < A,$$

in which

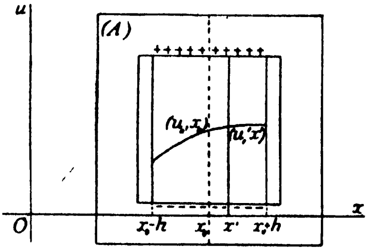
$$8)\qquad 0 < F_u(u, x).$$

And now consider the function 7) along the line $x = x_0$:

$$9)\qquad F(u, x_0).$$

This is a function of the single argument, $u$, which vanishes for $u = u_0$ and has a positive derivative when

$$u_0 - A' < u < u_0 + A'.$$



Hence

$$10)\qquad 0 < F(u_0 + A', x_0), \qquad F(u_0 - A', x_0) < 0.$$

Secondly, consider the function of $x$:

$$F(u_0 + A', x).$$

For $x = x_0$ it is positive, and since it is continuous, it must remain positive in a certain neighborhood of this point:

$$11)\qquad 0 < F(u_0 + A', x)$$

when

$$x_0 - h_1 < x < x_0 + h_1, \qquad 0 < h_1 \leqq A'.$$

Similarly,

$$12)\qquad F(u_0 - A', x) < 0$$

when $\quad x_0 - h_2 < x < x_0 + h_2, \qquad 0 < h_2 \leqq A'.$

Let $h$ denote the smaller of the two numbers $h_1$, $h_2$.

It is now easy to give the proof of the existence of a solution of the equation

$$13) \qquad F(u, x) = 0,$$

for which $x_0 - h < x < x_0 + h$ and $u_0 - A' < u < u_0 + A'$.

Let $x = x'$ be chosen arbitrarily in the interval

$$14) \qquad x_0 - h < x < x_0 + h.$$

Consider the function $F(u, x)$ for this value:

$$F(u, x').$$

We have here a function of the single variable $u$:

$$u_0 - A' \leqq u \leqq u_0 + A',$$

which is negative when $u = u_0 - A'$ and is positive for $u = u_0 + A'$. Since it is continuous, it must vanish in this interval. Moreover, it cannot vanish but once. For, if it had two roots, $u' < u''$, then by Rolle's Theorem

$$F_u(U, x') = 0, \qquad u' < U < u''.$$

But this is in contradiction of 8).

We have thus established the existence of a root $u$ of Equation 13) for each $x$ of the interval 14), and

$$u_0 - A' < u < u_0 + A'$$

Moreover, we have shown that there is only one such root. Thus we are led to a function

$$15) \qquad u = \varphi(x),$$

defined in the interval 14), single-valued, and such that

$$16) \qquad u_0 - A' < \varphi(x) < u_0 + A'.$$

And this function satisfies Equation 13).

This completes the proof of the Existence Theorem. It remains to establish the continuity of $\varphi(x)$ and the existence of a derivative.

The next step, then, consists in showing that $\varphi(x)$ is continuous. First, it surely is continuous at the point $x = x_0$:

$$|\varphi(x) - \varphi(x_0)| < \epsilon, \qquad |x - x_0| < \delta.$$

For $A'$ can be chosen as small as we wish and hence, if $\epsilon$ is less than the above $A'$, a new $A' = \epsilon$ can be selected.

Secondly, let $x'$ be an arbitrary value of $x$ in 14), and let $u' = \varphi(x')$. Then all of the hypotheses of the Existence Theorem are fulfilled in the neighborhood of the point $(u', x')$. In particular, then, the solution must be continuous at the one point, $x = x'$. But the solution is unique, and so coincides with the function 15) near this point. This completes the proof of continuity of the function $\varphi(x)$.

*Remark.* Up to the present we have made no use of the hypothesis of the existence of the partial derivative $\partial F/\partial x$; nor have we used the continuity of $\partial F/\partial u$, except to justify 8). It is enough, then, to assume the existence of this latter derivative, and the inequality 8).

*Differentiation.* The proof of the existence of a derivative of the function $\varphi(x)$ is the same as is given in the Calculus. It is enough to consider the point $x = x_0$. We have, by the Law of the Mean:

$$F(u_0 + \Delta u, x_0 + \Delta x) = \Delta u \, F_u(u_0 + \theta\Delta u, x_0 + \theta\Delta x)$$

$$+ \Delta x \, F_x(u_0 + \theta\Delta u, x_0 + \theta\Delta x) = 0,$$

$$\frac{\Delta u}{\Delta x} = -\frac{F_x(u_0 + \theta\Delta u, \, x_0 + \theta\Delta x)}{F_u(u_0 + \theta\Delta u, \, x_0 + \theta\Delta x)}.$$

The division is possible for values of $\Delta x$ suitably restricted, since the corresponding values of $\Delta u$ are also numerically small, and $F_u(u, x)$ is continuous and different from 0 at $(u_0, x_0)$. Hence

$$\lim_{\Delta x = 0} \frac{\Delta u}{\Delta x} = -\frac{F_x(u_0, x_0)}{F_u(u_0, x_0)}.$$

### EXERCISES

1. Carry through the proof in the next case,

$$F(u, x, y) = 0.$$

First, write out the theorem in detail for this case. Illustrate the region $(A)$ by a cube with its vertices in the eight points $(u_0 \pm A, x_0 \pm A, y_0 \pm A)$. Draw the space figure with ruler and pencil or pen. Draw in the smaller cube $(A')$ in less prominent lines. Then put in the region

$$|u - u_0| \leqq A', \qquad |x - x_0| < h, \qquad |y - y_0| < h$$

in red pencil or ink, or other colored lines that will make it stand out.

2. Insert at each point of the proof carried through in the text a precise reference to the theorem used, and make a list of all such theorems, which have been illustrated by the present applications.

3. Show that the equation

$$u^3 + 2u + e^{u-x-y^2} = \cos(x - y + u)$$

defines a single-valued function

$$u = \varphi(x, y),$$

continuous for all values of the arguments, $x$ and $y$.

4. Give an analytic proof of the theorem in the general case, observing the geometric significance of each step in space of $n$ dimensions.

5. Assuming that the function $F(u, x, y, \cdots)$ has continuous derivatives of the second order, and satisfies the other conditions of the theorem, prove that the function $\varphi(x, y, \cdots)$ has continuous derivatives of the second order. Generalize.

6. Show that, under the conditions of the theorem, there exists a subregion

$$R: \qquad |u - u_0| \leqq B', \qquad |x - x_0| \leqq A'$$

and a positive constant $h'$ such that, if

$$F(u_1, x_1) = 0, \qquad (u_1, x_1) \text{ in } R,$$

the equation

$$F(u, x) = 0$$

admits a solution

$$u = \varphi(x; x_1)$$

defined throughout the interval

$$-h' < x - x_1 < h',$$

and having the properties of the function $\varphi(x)$ of the theorem.

**§11. Simultaneous Systems of Equations.** *Let*

$$1) \qquad F_i(u_1, \cdots, u_p; x_1, \cdots, x_n), \qquad i = 1, \cdots, p,$$

*together with its first partial derivatives, be continuous in the neighborhood of the point* $(b_1, \cdots, b_p; a_1, \cdots, a_n)$ *and vanish there:*

$$F_i(b_1, \cdots, b_p; a_1, \cdots, a_n) = 0, \qquad i = 1, \cdots, p.$$

*Let the Jacobian,*

$$J = \begin{vmatrix} \dfrac{\partial F_1}{\partial u_1} \cdots \dfrac{\partial F_1}{\partial u_p} \\ \cdots\cdots\cdots\cdots \\ \dfrac{\partial F_p}{\partial u_1} \cdots \dfrac{\partial F_p}{\partial u_p} \end{vmatrix}$$

*be different from* 0 *there:*

$$J(b_1, \cdots, b_p; a_1, \cdots, a_n) \neq 0.$$

*Then there exist $p$ functions:*

3) $$u_i = \varphi_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, p,$$

*continuous in the neighborhood of the point $(a_1, \cdots, a_n)$ and taking on the value $b_i$ there:*

$$b_i = \varphi_i(a_1, \cdots, a_n), \qquad i = 1, \cdots, p;$$

*and such that, when substituted in the functions $F_i$, they cause these to vanish identically:*

$$F_i[\varphi_1(x_1, \cdots, x_n), \cdots, \varphi_p(x_1, \cdots, x_n); x_1, \cdots, x_n] \equiv 0, \quad i = 1, \cdots, p.$$

*Furthermore, the only roots of the simultaneous system of equations*

4) $$F_i(u_1, \cdots, u_p; x_1, \cdots, x_n) = 0, \qquad i = 1, \cdots, p,$$

*which lie in the neighborhood of $(b_1, \cdots, b_p; a_1, \cdots, a_n)$ are those given by Equation* 3).

*Finally, the functions $\varphi_i(x_1, \cdots, x_n)$ possess continuous derivatives of the first order, given by the ordinary rules of the Calculus.*

Consider the simplest case: $p = 2$, $n = 1$:

5) $$F(u, v, x) = 0, \qquad \Phi(u, v, x) = 0;$$

$$J = \begin{vmatrix} \dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial v} \\ \dfrac{\partial \Phi}{\partial u} & \dfrac{\partial \Phi}{\partial v} \end{vmatrix} \neq 0,$$

and denote the point by $(u_0, v_0, x_0)$. We wish to show that the simultaneous solutions of Eqs. 5), which lie in the neighborhood of this point, are given by two functions,

6) $$u = \varphi(x), \qquad v = \psi(x),$$

each continuous in the neighborhood of $x = x_0$ and taking on the respective values $u_0, v_0$ in this point.

The functions $F_u$ and $F_v$ cannot both vanish in the point $(u_0, v_0, x_0)$, for otherwise $J$ would vanish there. Let

7) $$F_u(u_0, v_0, x_0) \neq 0.$$

Then the equation

8) $$F(u, v, x) = 0$$

can be solved for $u$ by the Existence Theorem of § 10:

9) $$u = \omega(v, x),$$

and moreover *all* the roots of 8) which lie in this neighborhood are given by 9). Thus Equation 9) is *equivalent* to the first equation 5) in the sense that the two equations:

$$F(u, v, x) = 0, \qquad u = \omega(v, x)$$

have the same roots in the neighborhood of $(u_0, v_0, x_0)$.

From this fact it appears that the roots of the simultaneous system 5) coincide with the roots of the simultaneous system:

10) $$u = \omega(v, x), \qquad \Phi(u, v, x) = 0.$$

A necessary condition for such a root is that

11) $$\Phi[\omega(v, x), v, x] = 0.$$

And conversely, any root $(v, x)$ of 11), lying in a suitable neighborhood of $(v_0, x_0)$, will lead to a value of $u$ through 9), lying in the neighborhood of $u = u_0$, the tripel $(u_0, v_0, x_0)$ being a root of 5).

It remains, then, to solve Equation 11). Let

$$G(v, x) \equiv \Phi[\omega(v, x), v, x].$$

Then

$$G_v(v, x) = \Phi_u \frac{\partial \omega}{\partial v} + \Phi_v.$$

From 8)

$$\frac{\partial \omega}{\partial v} = -\frac{F_v}{F_u},$$

and hence

$$G_v(v, x) = \frac{J}{F_u}.$$

Thus

$$G_v(v_0, x_0) = \frac{J(u_0, v_0, x_0)}{F_u(u_0, v_0, x_0)} \neq 0,$$

and moreover $G(v, x)$ is seen to fulfil all the other conditions of the Existence Theorem of § 10. Hence Equation 11) is equivalent to the equation

$$v = \psi(x),$$

where $\psi(x)$ is a function continuous in the neighborhood of the point $x = x_0$ and taking on the value $v_0$ there. Let

$$\varphi(x) = \omega[\psi(x), x].$$

Then the functions

$$u = \varphi(x), \qquad v = \psi(x)$$

fulfill all the conditions of the Existence Theorem we set out to prove.

The extension of the proof to the case $p = 2$, $n = n$ requires no modification. When $p > 2$, the method of mathematical induction can be used. The partial derivatives of $F_1$ cannot all be 0. Let

$$\frac{\partial F_1}{\partial u_1} \neq 0.$$

Then the equation

$$F_1(u_1, \cdots, u_p;\ x_1, \cdots, x_n) = 0$$

can be solved for $u_1$:

12) $$u_1 = \omega(u_2, \cdots, u_p;\ x_1, \cdots, x_n).$$

This equation, combined with the last $p - 1$ equations 4), is equivalent to the original system 4). And now a necessary condition for a simultaneous solution of 4) is, that $(u_2, \cdots, u_p;\ x_1, \cdots, x_n)$ be a solution of the $p - 1$ equations:

13) $$F_i[\omega(u_2, \cdots, u_p;\ x_1, \cdots, x_n), u_2, \cdots, u_p;\ x_1, \cdots, x_n] = 0,$$
$$i = 2, \cdots, p.$$

The Jacobian of this system is seen to have the value

$$\frac{J}{\partial F_1/\partial u_1}$$

and so does not vanish at the point $(b_1, \cdots, b_p;\ a_1, \cdots, a_n)$. Thus all the conditions of the Existence Theorem before us are seen to be fulfilled for the system of $p - 1$ equations 13).

Conversely, any solution of 13), suitably restricted as to neighborhood, leads through 12) to a solution of the original system 4), and this completes the proof.

*Remark.* In the foregoing proof we have made no use of the derivatives $\partial F_i/\partial x_j$. Hence these need not exist. They are not needed till the next step.

*Differentiation.* Returning to the case $p=2$, $n=1$ we can prove the existence of derivatives as follows. Let

$$\Delta u = \varphi(x_0+\Delta x)-\varphi(x_0), \qquad \Delta v = \psi(x_0+\Delta x)-\psi(x_0).$$

From the law of the mean,

$$F(u_0+\Delta u, v_0+\Delta v, x_0+\Delta x) = \Delta u\, F_u + \Delta v\, F_v + \Delta x\, F_x = 0,$$

$$\Phi(u_0+\Delta u, v_0+\Delta v, x_0+\Delta x) = \Delta u\, \Phi_u + \Delta v\, \Phi_v + \Delta x\, \Phi_x = 0,$$

where $F_u, \cdots, \Phi_u, \cdots$ are formed for mean values of the arguments, as

$$F_u(u_0+\theta\Delta u, v_0+\theta\Delta v, x_0+\theta\Delta x), \quad \text{etc.}$$

Since $J(u_0, v_0, x_0) \neq 0$, it follows that

$$\begin{vmatrix} F_u & F_v \\ \Phi_u & \Phi_v \end{vmatrix},$$

formed for these values, will not vanish if $\Delta x$ is suitably restricted. Hence

$$\frac{\Delta u}{\Delta x} = -\frac{\begin{vmatrix} F_x & F_v \\ \Phi_x & \Phi_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ \Phi_u & \Phi_v \end{vmatrix}}, \qquad \frac{\Delta v}{\Delta x} = -\frac{\begin{vmatrix} F_u & F_x \\ \Phi_u & \Phi_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ \Phi_u & \Phi_v \end{vmatrix}},$$

formed for the above mean values. It is obvious that the right-hand sides approach limits as $\Delta x$ approaches 0, and hence $\varphi(x)$, $\psi(x)$ possess derivatives given by the ordinary rules of the Calculus.

## § 12. The Inverse of a Transformation.

*Let a transformation be given:*

$$1) \qquad y_i = f_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, n,$$

*where $f_i(x_1, \cdots, x_n)$, together with its derivatives of the first order, is single-valued and continuous in the neighborhood of the point $(a_1, \cdots, a_n)$, and let*

$$b_i = f_i(a_1, \cdots, a_n), \qquad i = 1, \cdots, n,$$

*Let the Jacobian:*

$$2)\qquad \frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)}$$

*be different from* 0 *in this point. Then the equations* 1) *admit a solution of the form:*

$$3)\qquad x_i = \varphi_i(y_1, \cdots, y_n), \qquad i = 1, \cdots, n,$$

*where* $\varphi_i(y_1, \cdots, y_n)$ *is single-valued and continuous in the neighborhood of the point* $(b_1, \cdots, b_n)$, *having continuous derivatives of the first order with non-vanishing Jacobian.*

For, form the equations:

$$F_i(x_1, \cdots, x_n;\, y_1, \cdots, y_n) \equiv f_i(x_1, \cdots, x_n) - y_i = 0,$$
$$i = 1, \cdots, n.$$

Identify these functions $F_i$ with the $F_i$ of the theorem of § 10, where $p = n$, where $u_i$ is replaced by $x_i$, and where the former $x_i$ is replaced by $y_i$. Then all the hypotheses of the former theorem are fulfilled, and the conclusion is a proof of the theorem in hand.

In case the Jacobian 2) vanishes, a single-valued continuous inverse 3) may still be possible; witness the example:

$$y = x^3, \qquad x = y^{\frac{1}{3}}.$$

But the derivatives of the inverse functions $\varphi$ cannot be continuous. For it is a property of Jacobians that*

$$\frac{\partial(u_1, \cdots, u_n)}{\partial(x_1, \cdots, x_n)} = \frac{\partial(u_1, \cdots, u_n)}{\partial(y_1, \cdots, y_n)}\,\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)},$$

and hence, in particular, in the case of the transformation 1):

$$\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)}\,\frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)} = 1.$$

If, however, the Jacobian 2) vanishes identically, a single-valued inverse is never possible; cf. § 15.

**§ 13. Identical Vanishing of the Jacobian.** Let $f(u, v)$, $\varphi(u, v)$ be two functions which, together with their first derivatives, are continuous in the neighborhood of a point $(u_0, v_0)$, and let their Jacobian vanish identically:

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* Cf. Jordan, *Cours d'analyse*, vol. 1 (1893) p. 89.

1)
$$\frac{\partial(f, \varphi)}{\partial(u, v)} \equiv 0.$$

Then $f$ and $\varphi$ are connected by a functional relation:

2)
$$\Omega\,[f(u, v),\ \varphi(u, v)] \equiv 0.$$

For, let

3)
$$x = f(u, v), \qquad y = \varphi(u, v).$$

If $f_u$, $f_v$ both vanish identically, then $f(u, v) = \text{const.}$ and we are through. Assume, then, that

$$\frac{\partial f}{\partial u} = f_u(u, v) \not\equiv 0.$$

Let $(u_1, v_1)$ be a point of the above neighborhood, in which

4)
$$f_u(u_1, v_1) \neq 0.$$

Then it is possible to solve the first of the equations 3) for $u$ in the neighborhood of this point:

5)
$$u = \omega(v, x).$$

Let us substitute this value in the second equation 3):

6)
$$y = \varphi[\omega(v, x), v].$$

The function on the right does not depend on $v$:

7)
$$\frac{\partial y}{\partial v} = 0.$$

For,

$$\frac{\partial \varphi[\omega(v, x), x]}{\partial v} = \varphi_u \frac{\partial \omega}{\partial v} + \varphi_v,$$

$$\frac{\partial \omega}{\partial v} = -\frac{f_v}{f_u},$$

and from 1) it follows that 7) is true. Thus

$$y = \Psi(x)$$

is true for all values of $x$ and $y$ given by 3), or

$$\varphi(u, v) \equiv \Psi[f(u, v)],$$

and that is what we set out to prove. The generalization is immediate:—

THEOREM. *Let*

$$8) \qquad f_i(u_1, \cdots, u_p; x_1, \cdots, x_n), \qquad i = 1, \cdots, p,$$

*be continuous, together with their first derivatives with respect to* $u_1, \cdots, u_p$, *in the neighborhood of a point* $(b_1, \cdots, b_p; a_1, \cdots, a_n)$, *and let the Jacobian*

$$9) \qquad \frac{\partial(f_1, \cdots, f_p)}{\partial(u_1, \cdots, u_p)} \equiv 0$$

*in all* $p + n$ *arguments. Then these* $p$ *functions are connected by an identical relation of the form:*

$$10) \qquad \Omega(f_1, \cdots, f_p) \equiv 0.$$

*More precisely, let*

$$y_i = f_i(u, x), \qquad i = 1, \cdots, p,$$

$$c_i = f_i(b, a).$$

*Let* $(b', a')$ *be a suitably chosen point of the above neighborhood, arbitrarily near to* $(b, a)$ *and let*

$$c_i' = f_i(b', a').$$

*Then there exists a function* $\Omega(y_1, \cdots, y_p)$, *continuous together with its first derivatives in the neighborhood of the point* $(y) = (c')$ *and vanishing there, but having at least one first derivative different from* 0 *there, and such that* 10) *is true for all* $(u, x)$ *in the neighborhood of* $(b', a')$.

Although $(b', a')$ can be taken arbitrarily near to $(b, a)$ there is no reason to assume that it can be made to coincide with this point. Certainly, in the corresponding case of analytic functions of several complex variables, this is not true.*

A generalization of the foregoing theorem for the case that the matrix of the determinant of the Jacobian 9) is of order less than $p - 1$ is given in the *Funktionentheorie*, l. c. § 23.

**14. Solutions in the Large.** The theorems of §§ 10-13 relate to solutions in a restricted region, the extent of which is not given explicitly at the outset, but is contained implicitly in the hypotheses of the theorems. Thus the results hold *in the small* (im

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* Cf. Osgood, *Funktionentheorie* $\text{II}_1$ Chap. 2, § 22.

Kleinen). There are no general methods for dealing with these questions in preassigned regions; i.e. *in the large* (im Grossen). Nevertheless there is a certain class of cases in which theorems in the small, when supplemented by uniform properties and the covering theorem of Chap. III, § 11, do lead to results in the large.

Consider, for example, the theorem of § 10. Let us add to the hypotheses the requirement that

$$1)\qquad 0 < \frac{\partial F}{\partial u}$$

in $(A)$. Consider a subregion

$$(A')\quad | u - u_0 | \leqq A', \quad | x - x_0 | \leqq A', \quad 0 < A' < A.$$

Let $(u_1, x_1)$ be any interior point of $(A')$. Then there passes through $(u_1, x_1)$ a solution of the equation

$$2)\qquad F(u, x) = 0,$$

namely,

$$3)\qquad u = \Phi(x),$$

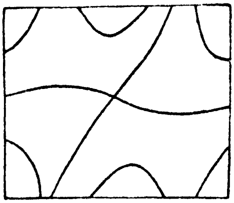
where the curve represented by 3) meets the boundary of $(A')$ in two points, the function $\Phi(x)$ being single-valued and continuous in a certain interval

$$\xi_1 \leqq x \leqq \xi_2$$

where

$$x_0 - A' \leqq \xi_1 < \xi_2 \leqq x_0 + A'.$$

There are obviously eight possibilities illustrated in the accompanying diagram.



The proof is as follows. According to § 10, Ex. 6 there exists a positive constant $h_1$ such that, if $(u_1, x_1)$ is a root of 2) lying in $(A')$, then a solution of 2) is given by the equation

$$4)\qquad u = \varphi(x)$$

where $\qquad x_1 - h_1 \leqq x \leqq x_1 + h_1.$

Thus starting with an arbitrary root $(u_1, x_1)$ of 2), which lies inside of $(A')$, we can proceed a distance of $h_1$ forward, and also a distance of $h_1$ backward. If the curve 4) still lies within $(A')$, we can now apply the equation 4) to an end-point of the arc already obtained, thus continuing the function $\varphi(x)$, and then repeat the process. Since we make progress each time by a distance $h_1$ along the axis of $x$, after a finite number of steps we must reach the boundary of $(A')$.

# Chapter V

## Uniform Convergence

**§ 1. Series of Functions.** Consider a series of functions:

1) $$u_1(x) + u_2(x) + \cdots$$

Let each term be continuous in the closed interval

$$a \leqq x \leqq b,$$

and let the series converge in each point of the interval. Denote the value of the series by $f(x)$; then

2) $$f(x) = u_1(x) + u_2(x) + \cdots$$

It is natural to think of the limiting function as continuous — partly from experience, for the power series we have met in the Calculus and used for computation, represent continuous functions; partly because the approximation curves,

3) $$y = s_n(x),$$

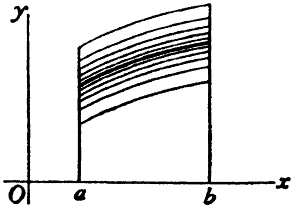
where

4) $$s_n(x) = u_1(x) + \cdots + u_n(x),$$

are all continuous and so would seem necessarily to approach the limiting function

5) $$y = f(x)$$

in the manner indicated in the figure.



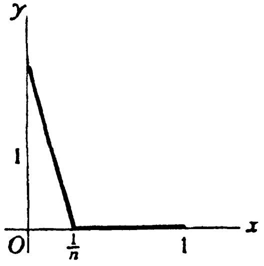
More fully analyzed this assumption consists of two parts: — first, that the limiting locus is a continuous curve; and secondly that, if this curve be embedded in ever so thin a strip, all the later approximation curves will come to lie within this strip.

Both parts of this assumption, however, are wrong, as is shown by the following examples.

*Example* 1. Let $s_n(x)$ be defined as suggested by the accompanying figure, namely[*]

$$s_n(x) = 1 - nx, \qquad 0 \leqq x \leqq \frac{1}{n};$$

$$s_n(x) = 0, \qquad \frac{1}{n} < x \leqq 1;$$



Here,

$$\lim_{n=\infty} s_n(x) = 0, \qquad 0 < x \leqq 1;$$

$$\lim_{n=\infty} s_n(0) = 1.$$

Thus we have an example of a convergent series of continuous functions which converges toward a discontinuous function,

$$f(x) = 0, \qquad 0 < x \leqq 1;$$

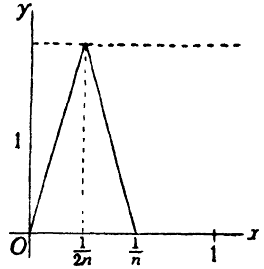
$$f(0) = 1.$$

*Example* 2. Let $s_n(x)$ be defined as suggested by this figure:

$$s_n(x) = 2nx, \qquad 0 \leqq x \leqq \frac{1}{2n};$$

$$s_n(x) = 2 - 2nx, \qquad \frac{1}{2n} < x \leqq \frac{1}{n};$$

$$s_n(x) = 0, \qquad \frac{1}{n} < x \leqq 1.$$



Here,

$$\lim_{n=\infty} s_n(x) = 0, \qquad 0 \leqq x \leqq 1,$$

and the limiting function,

$$f(x) = 0, \qquad 0 \leqq x \leqq 1,$$

is continuous. We have, then, it is true, a convergent series of continuous functions representing a continuous function. But the con-

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[*] We point out again that not only does an infinite series determine the sum of its first $n$ terms, but conversely any variable $s_n$ has corresponding to it an infinite series for which it is the sum of the first $n$ terms; namely, the series:

$$u_1 = s_1, \qquad u_2 = s_2 - s_1, \qquad \cdots, \qquad u_n = s_n - s_{n-1}.$$

It is, therefore, immaterial whether we think of the series as given by the $u_n$ or the $s_n$.

vergence is not like that represented in the figure of p. 132. If a small strip be constructed about the limiting locus,

$$y = f(x) = 0, \qquad 0 \leqq x \leqq 1,$$

the later approximation curves, no matter how great $n$, will fail to remain within this strip.

If, then, we wish to secure the kind of convergence suggested by the figure of p. 132, a further restriction is needed, and this leads us to *uniform convergence*, defined in the next paragraph.

### § 2. Uniform Convergence. *Definition.* Let

1) $$u_1(x) + u_2(x) + \cdots$$

be a series whose terms are defined in the points of an arbitrary infinite point set, $A$. The series is said to *converge uniformly* if, to a positive number $\epsilon$ chosen at pleasure there corresponds a natural number, $m$, *independent of* $x$, such that

$$|s_{n'}(x) - s_n(x)| < \epsilon, \qquad m \leqq n, n'.$$

Here,

$$s_n(x) = u_1(x) + \cdots + u_n(x),$$

and $x$ is any point in $A$.

Two consequences of the definition are expressed in the following theorems. The property of a series embodied in either theorem might have been chosen as the definition, and then the other theorem, and the definition actually laid down, would form the two complementary theorems.

THEOREM 1. *A necessary and sufficient condition for the uniform convergence of the series* 1) *is that, to a positive number* $\epsilon$ *chosen at pleasure, there correspond a natural number* $m$, *independent of* $x$, *such that*

$$|s_{m+p}(x) - s_m(x)| < \epsilon, \qquad p = 1, 2, 3, \cdots.$$

If the series 1) converges, let the remainder be denoted by $r_n(x)$:

2) $$f(x) = s_n(x) + r_n(x).$$

THEOREM 2. *A necessary condition for the uniform convergence of the series* 1) *is that, to a positive number* $\epsilon$ *chosen at pleasure, there correspond a natural number* $m$, *independent of* $x$, *such that*

$$|r_n(x)| < \epsilon, \qquad m \leqq n.$$

*If the series* 1) *converges, then this condition is conversely sufficient.*

The proofs of these theorems are immediate, since the student is now thoroughly familiar with what is meat by a *necessary* condition, and what is meant by a *sufficient* condition.

*Example*: The geometric series

$$1) \qquad 1 + x + x^2 + \cdots$$

converges uniformly in any interval $(a, b)$ which together with its end points lies *within* the interval $(-1, 1)$:

$$-1 < a < b < 1.$$

For, a number $h$ can be found such that

$$|a|, |b| < h < 1.$$

In the interval

$$-h \leqq x \leqq h$$

the remainder of the series,

$$r_n(x) = \frac{x^n}{1-x},$$

obviously satisfies the inequality:

$$\left|\frac{x_n}{1-x}\right| \leqq \frac{h^n}{1-h}.$$

Now, choose $m$ so that

$$\frac{h^m}{1-h} < \epsilon.$$

Then

$$|r_n(x)| < \epsilon, \qquad m \leqq n, \qquad q.e.d.$$

Observe, however, that the series 1) does *not* converge uniformly in the interval

$$2) \qquad -1 < x < 1.$$

If it did; i.e. if

$$|s_{n'}(x) - s_n(x)| < \epsilon, \qquad m \leqq n,$$

then it would follow, on setting $n' = n + 1$, that

$$|x^n| < \epsilon, \qquad m \leqq n.$$

But $m$ cannot be so chosen that

$$|x^m| < \epsilon$$

for all points of the interval 2), since

$$\lim_{x=1^-} x^m = 1.$$

The series 1) does, however, converge absolutely in the interval 2). We see, then, that absolute convergence does not insure uniform convergence.

EXERCISE

Show that the series

$$(1 - x) + (x^2 - x^3) + (x^4 - x^5) + \cdots$$

converges absolutely in the closed interval

$$0 \leqq x \leqq 1.$$

Observe that the terms are all $\geqq 0$.

Prove that the series converges uniformly in an arbitrary interval

$$0 \leqq x \leqq h, \qquad 0 < h < 1;$$

but that it does not converge uniformly in the interval $0 \leqq x \leqq 1$.

Plot accurately the first four approximation curves, using different colors — green, yellow, blue, red. Represent the limiting locus by a firm, black graph.

**§3. Weierstrass's *M*-Test.** A sufficient condition for the uniform convergence of a series is the following.

THE $M$-TEST. *The series* 1) *of* §2:

$$u_1(x) + u_2(x) + \cdots,$$

*converges uniformly in the point set $A$ if a convergent series of positive (or zero) constants*

$$M_1 + M_2 + \cdots, \qquad 0 \leqq M_n,$$

*can be found such that*

$$|u_n(x)| \leqq M_n, \qquad \mu \leqq n,$$

*where $\mu$ is a number independent of $x$.*

*Proof.* From the convergence of the $M$-series follows that to an arbitrary $\epsilon > 0$ corresponds an $m \geqq \mu$ such that

$$M_{n+1} + \cdots + M_{n'} < \epsilon, \qquad m \leqq n, n'.$$

Since

$$s_{n'}(x) - s_n(x) = u_{n+1}(x) + \cdots + u_{n'}(x),$$

we see that

$$|s_{n'}(x) - s_n(x)| \leqq |u_{n+1}(x)| + \cdots + |u_{n'}(x)|$$

$$\leqq M_{n+1} + \cdots + M_{n'}$$

and hence

$$|s_{n'}(x) - s_n(x)| < \epsilon, \qquad m \leqq n,\ n'.$$

This proves the theorem.

*Example.* Consider the series

$$\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \cdots.$$

The series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

is known to converge. Set

$$M_n = \frac{1}{n^2}.$$

Then the $M$-Test shows that the series converges uniformly for all values of $x$.

## EXERCISES

1. Show that the series

$$1 + \frac{1}{2} k^2 \sin^2 \varphi + \frac{1 \cdot 3}{2 \cdot 4} k^4 \sin^4 \varphi + \cdots,$$

where $k$ is a positive constant $< 1$, converges uniformly for all values of $\varphi$.

2. Does the series

$$\sum_{n=1}^{\infty} \frac{2x}{n^2 - x^2}$$

converge uniformly in the interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$? In the interval $-1 < x < 1$? Why?

3. Prove that the series

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges uniformly for all values of $x$ if the series

$$\sum_{n=1}^{\infty} a_n, \qquad \sum_{n=1}^{\infty} b_n$$

both converge absolutely.

4. Show that the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

converges uniformly in any bounded interval, but in no unbounded interval.

**§ 4. Continuity.** THEOREM, *Let the terms of the series*

$$u_1(x) + u_2(x) + \cdots$$

*be continuous in the closed interval*

$$a \leqq x \leqq b,$$

*and let the series converge uniformly in this interval. Then the function $f(x)$ defined by the series*

$$f(x) = u_1(x) + u_2(x) + \cdots$$

*is continuous.*

Let $x_0$ be an arbitrary point of the interval. We wish to prove that to an $\epsilon > 0$ chosen at pleasure corresponds a $\delta > 0$ such that

1) $$| f(x) - f(x_0) | < \epsilon, \qquad | x - x_0 | < \delta.$$

By hypothesis, to an arbitrary $\epsilon' > 0$ corresponds an $m$ independent of $x$ such that

2) $$| r_n(x) | < \epsilon, \qquad m \leqq n,$$

for all points $x$ of the interval. In particular, then, since

$$f(x) = s_n(x) + r_n(x),$$

we see from 2), on setting $n = m$ and writing the resulting inequality first for $x_0$, then for $x$, that

3) $$| f(x_0) - s_m(x_0) | < \epsilon'.$$

4) $$| f(x) - s_m(x) | < \epsilon'$$

On the other hand, $m$ being now a constant, we infer from the continuity of the $u_n(x)$ the continuity of the sum, $s_m(x)$, of a fixed number of them. Hence

5) $$| s_m(x) - s_m(x_0) | < \epsilon', \qquad | x - x_0 | < \delta.$$

Combining the inequalities 3), 4), 5) according to Chap. II, § 10 we obtain the relation:

$$6) \qquad |f(x)-f(x_0)| < 3\epsilon', \qquad |x-x_0| < \delta.$$

If, then, we choose $\epsilon'$, which is at our disposal, equal to $\frac{1}{3}\epsilon$, the relation 6) becomes the relation 1) which we wished to establish*.

### Exercises

1. Let

$$u_1(x) + u_2(x) + \cdots$$

be a series whose terms are defined in the points of an arbitrary infinite point set $M$, and let the series converge uniformly in $M$. Let $x = c$ be a cluster point of $M$, — regardless of whether $c$ is a point of $M$ or not. (In particular, $c$ may be the point $\infty$.) Let each term approach a limit as $x$ approaches $c$:

$$\lim_{x=c} u_n(x) = U_n.$$

Then

i) The series of limits:

$$U_1 + U_2 + \cdots$$

converges.

---

* It is well worth the student's time to study this theorem and its proof geometrically, interpreting the condition of uniform convergence:

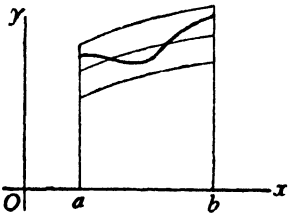
$$s_m(x) - \epsilon < s_n(x) < s_m(x) + \epsilon$$

as meaning that all the later approximation curves lie in the strip bounded by the curves

$$y = s_m(x) - \epsilon, \qquad y = s_m(x) + \epsilon,$$

Then narrow the strip by choosing a new $\epsilon' < \epsilon$, and show eometrically what happens.

From a geometric appreciation of what is going on analytically it is possible to *see* that the limiting locus *must* be a continuous curve, and then to establish this result rigorously by analysis; i.e. by means of the inequalities which define continuity and uniform convergence.



The details are given in the author's *Funktionentheorie*, vol. I, Chap. III, § 3; and also in the *Bull. Amer. Math. Soc.* ser. 2, vol. 3, Nov. 1896.

ii) The function $f(x)$ defined by the series,

$$f(x) = u_1(x) + u_2(x) + \cdots$$

converges:

$$\lim_{x=c} f(x) = A.$$

iii) The limit of the series, namely $A$; and the series of the limits, namely

$$B = U_1 + U_2 + \cdots,$$

are equal, or

$$A = B.$$

The result can be written compactly in the form:

$$\lim_{x=c} f(x) = \lim_{x=c} u_1(x) + \lim_{x=c} u_2(x) + \cdots.$$

2. When $m$ is a natural number,

$$\left(1 + \frac{1}{m}\right)^m = 1 + 1 + \frac{1 - \frac{1}{m}}{1\cdot 2} + \frac{\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)}{1\cdot 2\cdot 3}$$
$$+ \cdots \text{ to } m + 1 \text{ terms.}$$

Prove that

$$\left(1 + \frac{1}{m}\right)^m$$

approaches a limit when $m = \infty$, and that

$$\lim_{m=\infty} \left(1 + \frac{1}{m}\right)^m = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

3. Let the factors of the infinite product

$$\prod_{n=1}^{\infty} [1 + u_n(x)]$$

be continuous in the closed interval $a \leqq x \leqq b$. Let the series

$$u_1(x) + u_2(x) + \cdots$$

satisfy an $M$-test:

$$|u_n(x)| \leqq M_n, \qquad m \leqq n;$$

$\Sigma M_n$, convergent. Show that the product represents a continuous function, $f(x)$; and that

$$f(x) = \prod_{n=1}^{p} [1 + u_n(x)]\, \varphi(x),$$

where $p$ is fixed, and $\varphi(x)$ is continuous and does not vanish.

4. Prove that the infinite product

$$x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)$$

represents a continuous function, $f(x)$, for all values of $x$. Show that $f(x)$ has its roots in the points $x = 0, \pm 1, \pm 2, \cdots$, and that

$$\lim_{x=n}\frac{f(x)}{x-n} \text{ exists and } \neq 0,$$

$n$ being any integer.

## § 5. Power Series.

Consider the power series

$$1) \qquad a_0 + a_1x + a_2x^2 + \cdots$$

For a particular value of $x$ different from 0, $x' \neq 0$, let its terms be bounded:

$$|a_n x'^n| \leqq G.$$

Set $|x'| = X$. Then

$$|a_n| \leqq G X^{-n}.$$

For an arbitrary $x$ such that $|x| < X$ the series 1) converges absolutely. For

$$|a_n x^n| = |a_n|\cdot|x|^n \leqq G\left(\frac{|x|}{X}\right)^n.$$

The series whose general term is this last expression is a convergent geometric series, and the theorem is proved. Let the result be restated as

THEOREM 1. *If the terms of the power series*

$$a_0 + a_1x + a_2x^2 + \cdots$$

*and bounded for a particular* $x' \neq 0$, *the series converges absolutely for all* $x$ *in the interval* $|x| < |x'|$.

A power series may converge for all values of $x$, like

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

or it may diverge for all $x \neq 0$, like

$$x + 2!\,x^2 + 3!\,x^3 + \cdots.$$

In all other cases the points for which the series converges constitute an interval:

$$-R < x < R,$$

together with one, both, or neither of the end points.

For, let $\xi$ be a positive number for which the series converges. The point set $\{\xi\}$ is bounded, since by hypothesis the series diverges for some $x = x''$. It cannot, therefore, converge for a $\xi > |x''|$, because of Theorem 1.

Let $R$ be the upper limit of the point set $\{\xi\}$. Then this is the $R$ of the theorem. We will formulate the result as

THEOREM 2. *The domain of convergence of a power series which converges for some, but not all values of* $x \neq 0$, *is an interval*

$$-R < x < R,$$

*to which one or both of the end points may still have to be adjoined.*

For the interior points of the interval the series converges absolutely. For the end points all conceivable behaviors occur.

The term "convergent power series" is used by some writers to describe a power series which converges for values of $x \neq 0$.

The interval $(-R, R)$ of Theorem 2; or in case the power series converges for all values of $x$, the point set $-\infty < x < \infty$, is called the *interval of convergence.*

So much for the plain convergence of a power series. We turn now to the question of uniform convergence.

THEOREM 3. *A power series converges uniformly throughout any subinterval* $(a, b)$ *which together with its end points lies within the interval of convergence:*

$$-R < a < b < R.$$

For it is possible to choose a positive number $h$ so that

$$|a|,\ |b| < h < R.$$

In the point $x = h$ the power series will converge absolutely:

$$|a_0| + |a_1|h + |a_2|h^2 + \cdots,$$

a convergent series. If, then, we set

$$|a_n| h^n = M_n,$$

the conditions of the $M$-test will be fulfilled in the interval

$$-h \leqq x \leqq h,$$

and so the theorem is proved.

It has already been pointed out in § 2 that a power series does not in general converge uniformly in its interval of convergence. In certain cases it may do so, as:

$$\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots, \qquad -1 \leqq x \leqq 1.$$

The terms of a power series are continuous for all values of $x$. It follows, then, by the aid of § 4 and the theorem just proven that a power series represents a continuous function throughout any subinterval $(a, b)$ of its interval of convergence. This is not the same thing as saying that it represents a continuous function throughout its whole domain of convergence; but it is true, nevertheless, that it does.

THEOREM 4. *A power series represents a continuous function throughout its whole domain of convergence.*

Let $x'$ be any interior point of the interval of convergence, $(-R, R)$. It is then possible to choose the interval $(a, b)$ so as to include $x'$ in its interior. Hence the function defined or represented by the power series will be continuous in $x'$. But $x'$ was *any* interior point, and so the theorem is proved for *all* such points.

But this is not the complete theorem. Consider, for example, the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \qquad -1 < x \leqq 1.$$

We have not shown that the function is continuous in the point $x = 1$ of the domain of convergence, nor can we show it by the $M$-test, since the series

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

does not converge absolutely.

In a remarkable paper, on the Binomial Series, Abel supplied precisely the proof that is needed here. It depends on a lemma which he introduced and which is of importance in many branches of analysis; cf. § 6. First, however, one last theorem on power series.

THEOREM 5. *If a power series vanishes identically in the neighborhood of the origin, each coefficient is* 0.

Let

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

be a power series which converges in a certain region, $-R < x < R$, and let it vanish for all values of $x$ for which $-h < x < h$, where $0 < h \leqq R$:

$$0 = a_0 + a_1 x + a_2 x^2 + \cdots$$

Set $x = 0$. Hence $a_0 = 0$. And now

$$0 = a_1 x + a_2 x^2 + \cdots = x(a_1 + a_2 x + \cdots),$$

where this last power series converges, $-R < x < R$; Chap. VI, § 1. It follows, then, that

$$0 = a_1 + a_2 x + \cdots, \qquad 0 < |x| < h.$$

This series represents a continuous function, by Theorem 4. Let $x$ approach 0 as its limit. Then

$$0 = a_1.$$

On repreating the reasoning it appears that $a_m = 0$, and this completes the proof.

COROLLARY. *If two power series,*

$$a_0 + a_1 x + a_2 x^2 + \cdots,$$

$$b_0 + b_1 x + b_2 x^2 + \cdots,$$

*have the same value at all points in the neighborhood of the origin, then corresponding coefficients are equal:*

$$a_n = b_n, \qquad n = 0, 1, 2, \cdots.$$

For, their difference can be represented as a power series,

$$(a_0 - b_0) + (a_1 - b_1) x + (a_2 - b_2) x^2 + \cdots,$$

which vanishes identically in the neighborhood of the origin.

### EXERCISES

1. If

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

converges in the interval $(-R, R)$, show that

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

converges in the same interval.

2. If a power series converges in a certain interval and vanishes at the origin, but does not vanish identically, show that it has no second root in the neighborhood of the origin.

3. In the older books Theorem 5 was often proved as follows, and this proof was copied in the school algebras. Set $x = 0$. Hence $a_0 = 0$, and so

$$0 = a_1 x + a_2 x^2 + \cdots.$$

Divide through by $x$, thus getting:

$$0 = a_1 + a_2 x^2 + \cdots.$$

Now set $x = 0$ again; thus $a_1 = 0$. And so on. What is wrong in this proof?

4. After the error mentioned in Question 3 was pointed out, the writer of the school algebra modified his proof by saying: "In the equation

$$0 = a_1 + a_2 x + \cdots,$$

let $x$ approach 0 as its limit. Then $0 = a_1$." What assumption was he making here?

5. Given a power series:

$$a_0 + a_1 x + a_2 x^2 + \cdots.$$

If

$$\sqrt[n]{|a_n|} < \frac{1}{h}, \qquad 0 < h, \qquad m \leqq n,$$

show that the series converges when $-h < x < h$.

6. Show that the power series of Question 5 will converge for all values of $x$ if and only if

$$\lim_{n=\infty} \sqrt[n]{|a_n|} = 0.$$

**§6. Abel's Lemma.**[*] *Let*

1) $$s_k = u_1 + u_2 + \cdots + u_k$$

*and let*

2) $$a \leqq s_k \leqq A, \qquad k = 1, 2, \cdots, n.$$

*Let*

3) $$\epsilon_1 \geqq \epsilon_2 \geqq \cdots \geqq \epsilon_n \geqq 0.$$

*Then*

4) $$\epsilon_1 a \leqq \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots + \epsilon_n u_n \leqq \epsilon_1 A.$$

*Proof.* Since

$$u_1 = s_1, \quad u_2 = s_2 - s_1, \quad \cdots, \quad u_n = s_n - s_{n-1},$$

we can write:

$$\epsilon_1 u_1 + \epsilon_2 u_2 + \cdots + \epsilon_n u_n =$$

$$\epsilon_1 s_1 + \epsilon_2 (s_2 - s_1) + \cdots + \epsilon_n (s_n - s_{n-1}) =$$

$$(\epsilon_1 - \epsilon_2) s_1 + (\epsilon_2 - \epsilon_3) s_2 + \cdots + (\epsilon_{n-1} - \epsilon_n) s_{n-1} + \epsilon_n s_n.$$

Multiply the $k$-th relation 2) through by $\epsilon_k - \epsilon_{k+1}$, $k = 1, 2, \cdots, n - 1$; the $n$-th by $\epsilon_n$, and add. The result is the relation 4) which the Lemma calls for.

*Application.* Let the power series 1), § 5 converge for $x = r \neq 0$. Then it converges uniformly in the closed interval from $x = 0$ to $x = r$.

We wish to prove that, to an arbitrary $\epsilon > 0$, corresponds an $m$, independent of $x$, such that

5) $$| a_{m+1} x^{m+1} + \cdots + a_{m+p} x^{m+p} | < \epsilon, \qquad p = 1, 2, \cdots,$$

where $0 \leqq x \leqq r$ or else $r \leqq x \leqq 0$.

From the hypothesis of convergence for $x = r$ follows that

$$-\epsilon' < a_{m+1} r^{m+1} + \cdots + a_{m+p} x^{m+p} < \epsilon', \qquad p = 1, 2, \cdots.$$

Now choose $u_k = a_{m+k} r^{m+k}$; $\epsilon_k = \left(\frac{x}{r}\right)^{m+k}$. Then

$$-\left(\frac{x}{r}\right)^{m+1} \epsilon' \leqq \left(\frac{x}{r}\right)^{m+1} a_{m+1} r^{m+1} + \cdots$$

$$+ \left(\frac{x}{r}\right)^{m+p} a_{m+p} r^{m+p} \leqq \left(\frac{x}{r}\right)^{m+1} \epsilon',$$

---

[*] *Journal für Mathematik*, vol 1 (1826) p. 311.

or, since $$0 \leqq \left(\frac{x}{r}\right)^{m+1} \leqq 1,$$

6) $$-\epsilon' \leqq a_{m+1}\,x^{m+1} + \cdots + a_{m+p}\,x^{m+p} \leqq \epsilon'.$$

If, then, we take $0 < \epsilon' < \epsilon$, 5) will follow from 6).

Thus the proof of Theorem 4, § 5 is now complete.

### EXERCISES

1. It can be shown by mathematical induction that

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\cos \frac{1}{2}x - \cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x},$$
$$x \neq 2k\pi.$$

Prove that the series

$$\frac{\sin x}{1} + \frac{\sin 2x}{2} + \cdots$$

converges uniformly in any interval $(a, b)$ which together with its end points lies within the interval $0 < x < 2\pi$.

2. The same for the series

$$c_1 \sin x + c_2 \sin 2x + \cdots,$$

where

$$c_1 \geqq c_2 \geqq \cdots, \qquad \lim_{n=\infty} c_n = 0.$$

3. Show that

$$\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\cos nx - \cos(n+1)x}{2(1-\cos x)}.$$

4. What can you say about the uniform convergence of the series

a) $$\frac{\cos x}{1} + \frac{\cos 2x}{2} + \cdots ?$$

b) $$c_1 \cos x + c_2 \cos 2x + \cdots ?$$

**§ 7. The Binomial Series.** In the noted paper cited in § 6 Abel discussed the series

1) $$1 + mx + \frac{m(m-1)}{1\cdot 2}x^2 + \cdots,$$

making no assumptions about the function it may represent.

If $m$ is a natural number, the series breaks off with a finite number of terms and so converges for all values of $x$. Its value is, by the binomial theorem,

$$2) \qquad (1+x)^m.$$

For any other value of $m$ the series converges in the interval

$$-1 < x < 1$$

and diverges when $|x| > 1$. Denote its value by $f(m, x)$:

$$3) \quad f(m,x) = 1 + mx + \frac{m(m-1)}{1\cdot 2}x^2 + \cdots, \qquad -1 < x < 1.$$

Give $x$ such a value and hold it fast. Then

$$4) \qquad f(m+n, x) = f(m,x)f(n,x),$$

as we will now show.

Denote the binomial coefficient by $m_k$:

$$5) \qquad m_k = \frac{m(m-1)\cdots(m-k+1)}{1\cdot 2\cdots k}$$

Then the series that define the factors on the right of 4) can be written:

$$f(m,x) = 1 + m_1 x + m_2 x^2 + \cdots;$$
$$f(n,x) = 1 + n_1 x + n_2 x^2 + \cdots.$$

These series can be multiplied together by the theorem of Chap. VII, § 5:

$$6) \quad f(m,x)f(n,x) = 1 + (m_1 + n_1)x + (m_2 + m_1 n_1 + n_2)x^2 + \cdots$$

We wish to identify this latter series with the one for $f(m+n, x)$:

$$7) \qquad f(m+n,x) = 1 + (m+n)_1 x + (m+n)_2 x^2 + \cdots.$$

This can done expeditiously as follows. Observe that the coefficient of $x^k$ in 6) is a polynomial $G_k(m,n)$ of degree not higher than $k$ in each argument. The same is true of the series 7). Now, when $m$ and $n$ are natural numbers, Equation 4) is true, because of 2):

$$8) \qquad G_k(m,n) = (m+n)_k.$$

Hence 8) is true for all values of $m$ and $n$ by the following theorem.

*An Algebraic Theorem.* Let

$$a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$

be a polynomial which vanishes for $m+1$ distinct values of the argument: $x = \xi_0, \xi_1, \cdots, \xi_m$. Then each coefficient $a_k = 0$.

Suppose the theorem false. Let $a_\mu$ be the first coefficient that $\neq 0$. It surely will not be $a_m$. Now

$$a_\mu x^{m-\mu} + a_{\mu+1} x^{m-\mu-1} + \cdots + a_m = a_\mu (x - \xi_1) \cdots (x - \xi_\mu).$$

Set $x = \xi_0$, and this last expression must vanish. But no factor vanishes, and here is a contradiction.

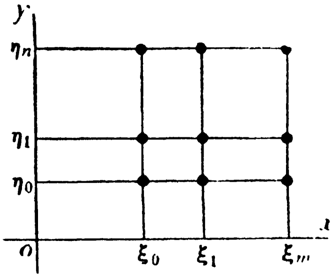
Secondly, let

$$G(x, y) = A_0(y)\, x^m + A_1(y)\, x^{m-1} + \cdots + A_m(y),$$

where

$$A_k(y) = b_0^{(k)} y^n + b_1^{\,k} y^{n-1} + \cdots + b_n^{(k)},$$

be a polynomial which vanishes in each point $(x, y) = (\xi_i, \eta_j)$, where $\xi_0, \xi_1, \cdots, \xi_m$ are $m+1$ distinct numbers, and likewise $\eta_0, \eta_1, \cdots, \eta_n$ are $n+1$ distinct numbers. Then each coefficient $b_l^{(k)} = 0$.



For, give to $y$ an arbitrary value $\eta_j$, and hold it fast. Then $G(x, \eta_j)$ vanishes for $\xi_0, \xi_1, \cdots, \xi_m$. Hence each $A_k$ vanishes:

$$A_k(\eta_j) = 0, \quad k = 0, 1, \cdots, m.$$

Hold $k$ fast and let $\eta_j$ run through the $n+1$ values $\eta_0, \eta_1, \cdots, \eta_n$. Thus we see that each $b_l^{(k} = 0$. The proof of the relation 4) is now complete.

The function $f(m, x)$ can be evaluated on the basis of the functional equation 4) and the continuity of the function inherent in its expression by means of the series. First, a digression

*On the Functional Equation:*[*]

$$F(x + y) = F(x)\, F(y). \tag{9}$$

Let

$$F(x), \qquad -\infty < x < \infty,$$

be a solution.

---

[*] Cauchy treated this and allied functional equations in the *Cours d'analyse algébrique* of 1821; Chap. 5.

i) If $F(x)$ vanishes for a single value of $x$:.

$$F(x_0) = 0,$$

then

$$F(x) \equiv 0.$$

For, set $y = x_0$. Then

$$F(x + x_0) = F(x)\, F(x_0) = 0$$

for all values of $x$.

ii) Excluding the case just considered, we see that

$$0 < F(x), \qquad -\infty < x < \infty.$$

For

$$F(x) = F\left(\frac{x}{2}\right) F\left(\frac{x}{2}\right).$$

iii) Again. excluding Case i), let $x = y = 0$. Then

$$F(0) = F(0)^2.$$

Since $F(0) \neq 0$, it follows that

10) $$F(0) = 1.$$

Let

11) $$F(1) = a.$$

Then

$$F(2) = F(1)\, F(1) = a^2,$$

$$F(3) = F(2)\, F(1) = a^3,$$

and, generally,

$$F(m) = a^m, \quad m = 0, 1, 2. \cdots.$$

Next, let $x = y = .\frac{1}{2}$ Then

$$F(1) = F(\tfrac{1}{2})\, F(\tfrac{1}{2}),$$

$$F(\tfrac{1}{2}) = a^{\frac{1}{2}}.$$

Since obviously

$$F(x_1 + x_2 \cdots + x_n) = F(x_1)\, F(x_2) \cdots F(x_n),$$

it follows that

12) $$F\left(\frac{1}{n}\right) = a^{\frac{1}{n}}.$$

That a positive $n$-th root of any positive number exists — and only one such root — was shown in Chap. II, § 10.

We now infer that

13) $$F\left(\frac{m}{n}\right) = a^{\frac{m}{n}}.$$

where $m$ and $n$ are any two natural numbers.

Let $y = -x$. Then

$$F(0) = F(x)\,F(-x),$$

14) $$F(-x) = \frac{1}{F(x)}.$$

Thus we arrive at the result that

15) $$F(\xi) = a^{\xi},$$

where $\xi$ is any commensurable number.

*Continuity.* Concerning the continuity of $F(x)$ we know nothing. We readily see, however, that if $F(x)$ is continuous for a single value, $x = x_0$, then $F(x)$ is continuous for all values of $x$. For,

$$F(x_0 + h) - F(x_0) = F(x_0)\,[\,F(h) - 1\,].$$

If the left hand side approaches 0 with $h$ for a single $x_0$, it follows that

$$\lim_{h=0} F(h) = 1.$$

Hence the left hand side approaches 0 for an arbitrary $x_0$.

*The Function* $f(m, x)$. Identifying the function $f(m, x)$ with $F(m)$ we see that

$$f(m, x) = (1 + x)^m,$$

or

16) $$(1 + x)^m = 1 + m\,x + \frac{m\,(m-1)}{1\cdot 2}\,x^2 + \cdots$$

for all rational values of $m$.

On the other hand the series 16) represents a continuous function of $m$ for all values of $m$. For, it converges uniformly in any bounded interval, and its terms are continuous. Let $M$ be any positive constant. Then

$$\left|\frac{m\,(m-1)\cdots(m-k+1)}{1\cdot 2\cdots k}\,x^k\right| \leqq$$

$$\frac{M\,(M+1)\cdots(M+k-1)}{1\cdot 2\cdots k}\,|\,x\,|^k = M_k, \qquad |\,m\,| \leqq M,$$

and the series

$$M_1 + M_2 + \cdots$$

is shown at once to converge. Hence $f(m, x)$ is continuous in the interval $(-M, M)$.

Let $m = m'$ be any value of $m$. Then $M$ can be so chosen that $m'$ will lie within the interval $(-M, M)$. Hence $f(m, x)$ is continuous in the point $m'$, and the theorem is proved.

*The Function* $a^x$. Let $a$ be any positive constant less than 2. Set

$$1 + x = a, \qquad x = a - 1.$$

Then

$$-1 < x < 1,$$

and this is a value of $x$ such as we have been considering.

If follows, then, that for such a value of $a$ there is a function of $m$, continuous for all values of $m$ and coinciding with

$$a^m$$

for all commensurable values of $m$. We define $a^m$ for irrational values of $m$ as equal to this function. Thus the function $a^m$, or $a^x$, is single-valued and continuous for all values of the exponent, $m$ or $x$, if $0 < a < 2$.

If $a \geqq 2$, let

$$a' = \frac{1}{a}.$$

Since

$$a'^m = \frac{1}{a^m}$$

when $m$ is rational, and since $a'^m$ is continuous and positive for all values of $m$, it follows that there exists a function continuous and positive for all values of $m$, and coinciding with $a^m$ when $m$ is rational. We define $a^m$ as equal to this function when $m$ is irrational.

Thus the function $a^x$ is defined and continuous for each positive value of $a$. Moreover,

$$a^{x+y} = a^x a^y.$$

On the other hand we have found the most general continuous solution of the functional equation 9):

$$F(x) = a^x, \qquad a = F(1); \quad \text{or} \quad F(x) \equiv 0.$$

*The Functions* $e^x$, $\log x$, $x^n$. With the aid of the results just obtained, together with the Exercises that follow, it is possible to develop systematically the theory of the functions

$$e^x, \qquad \log x, \qquad x^n.$$

The reader will find in profitable to make this study later. A better first approach to these functions, however, is through the integral; Chap. VI.

### EXERCISES

1. If $x$ is a constant numerically less than unity, show that

$$\lim_{m=0} \frac{(1+x)^m - 1}{m} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

2. If $F(x)$ is a continuous solution of the functional equation 9), show that

$$\lim_{h=0} \frac{F(h) - 1}{h}$$

exists.

3. Show that a continuous solution of the functional equation 9) has a derivative:

$$\frac{d\,F(x)}{dx} = c\,F(x),$$

where

$$c = \lim_{h\ 0} \frac{F(h) - 1}{h}.$$

4. Prove that

$$\lim_{\mu} \left(1 + \frac{1}{\mu}\right)^{\mu} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

when $\mu$ becomes infinite, passing through all real values numerically greater than 1.

5. Assuming the properties of the functions $e^x$, $\log x$, developed in Chap. VI, discuss the functional equation:

$$F(x+y) = F(x) + F(y).$$

Show that its most general continuous solution is:

$$F(x) = cx, \qquad -\infty < x < \infty,$$

where $c$ is a constant.

**§8. Integration of Series.** Let a convergent series of continuous functions be given:

$$1) \qquad f(x) = u_1(x) + u_2(x) + \cdots, \qquad a \leqq x \leqq b.$$

Can it be integrated term-by-term? i.e. will

$$2) \qquad \int_a^b f(x)\,dx = \int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx + \cdots$$

be a true equation?

Let us analyse this question in detail. There are really three questions here rolled into one: —

a) Will the series

$$3) \qquad \int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx + \cdots$$

converge?

b) Can the function $f(x)$ be integrated? i.e. does

$$4) \qquad \int_a^b f(x)\,dx$$

have a meaning?

c) If the answers to Questions a) and b) are both affirmative, will Equation 2) be true?

Consider the approximation curves,

$$5) \qquad y = s_n(x),$$

where

$$s_n(x) = u_1(x) + \cdots + u_n(x).$$

They are all continuous, and so

$$\int_a^b s_n(x)\,dx$$

means the area under the $n$-th curve.

Consider the series 3). Let

$$6) \qquad S_n = \int_a^b u_1(x)\,dx + \cdots + \int_a^b u_n(x)\,dx.$$

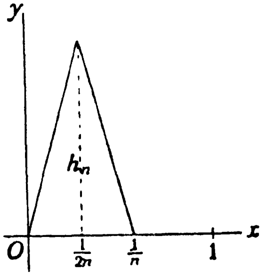
The series 3) converges if $S_n$ approaches a limit; and conversely. Now,

$$S_n = \int_a^b s_n(x)\,dx. \tag{7}$$

Hence Question a) is the question of whether the area under the curve 5) approaches a limit.

*Example.* Let

$$\begin{cases} s_n(x) = 2nh_n x, & 0 \leqq x \leqq \dfrac{1}{2n}; \\ s_n(x) = 2h_n - 2nh_n x, & \dfrac{1}{2n} < x \leqq \dfrac{1}{n}; \\ s_n(x) = 0, & \dfrac{1}{n} < x \leqq 1. \end{cases}$$



The series 1) converges, and

$$f(x) = 0, \qquad 0 \leqq x \leqq 1.$$

The value of $S_n$ is the area of the isosceles triangle, or

$$S_n = \frac{h_n}{2n}.$$

We have not yet said how $h_n$ shall vary with $n$. Suppose, for example, that

i) $h_n = n^2$. Then

$$S_n = \frac{n}{2}$$

and the series 3) diverges. So, naturally, there cannot be any question of integrating it term-by-term. — Suppose, however, that

ii) $h_n = \sqrt{n}$. Now,

$$S_n = \frac{1}{2\sqrt{n}}.$$

The series 3) converges, and its value is 0. On the other hand the integral 4) converges, and its value is 0, too. So here the series 1) can be integrated term-by-term. — Lastly, let

iii) $h_n = n$. Consequently

$$S_n = \tfrac{1}{2}.$$

The series 2) converges; but its value, $\frac{1}{2}$, is not equal to the integral 4), or 0.

We see, then, that even though each side of equation 2) may have a meaning, it does not follow that the equation is true.

A *sufficient* condition for integrating a series term-by-term is the following.

THEOREM. *Let the terms of the series*

$$u_1(x) + u_2(x) + \cdots$$

*be continuous in the closed interval*

$$a \leqq x \leqq b,$$

*and let the series converge uniformly in this interval. Then the series can be integrated term-by-term:*

$$f(x) = u_1(x) + u_2(x) + \cdots;$$

$$\int_a^b f(x)\,dx = \int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx + \cdots$$

*Proof.* The function $f(x)$ is continuous; Theorem, § 4. So the integral

8) $$\int_a^b f(x)\,dx$$

has a meaning. Next, let

$$f(x) = s_n(x) + r_n(x).$$

Then, the functions $s_n(x)$, $r_n(x)$ being continuous, we have:

$$\int_a^b f(x)\,dx - \int_a^b s_n(x)\,dx = \int_a^b r_n(x)\,dx.$$

Since the series converges uniformly,

$$|r_n(x)| < \epsilon, \qquad m \leqq n.$$

Hence

$$\left|\int_a^b r_n(x)\,dx\right| \leqq \int_a^b |r_n(x)|\,dx < (b-a)\,\epsilon, \qquad m \leqq n.$$

It follows, then, that

$$\left| \int_a^b f(x)\,dx - \int_a^b s_n(x)\,dx \right| < (b-a)\,\epsilon, \qquad m \leqq n,$$

and so i) the series of integrals converges; and ii) its value is the integral of the series, or 8).

### EXERCISES

1. Show that a power series can be integrated term-by-term. State this theorem precisely.

2. Prove:

$$\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \right.$$

$$\left.\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right) k^6 + \cdots\right], \quad 0 \leqq k < 1.$$

3. Prove:

$$\int_0^{\frac{\pi}{2}} \sqrt{1-e^2\sin^2\varphi}\; d\varphi = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{e^4}{3} - \right.$$

$$\left.\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{e^6}{5} - \cdots\right], \quad 0 \leqq e < 1.$$

4. Give a new example of a series of continuous functions which does not converge uniformly, and still can be integrated term-by-term.

5. In $\epsilon$-proofs like the above we have hitherto used two $\epsilon$'s — the $\epsilon$ of our adversary, and the $\epsilon'$ of our own. Throw the proof of the Theorem of the text into that form.

6. Let

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

be a power series which converges when $0 \leqq x < h$, but diverges for $x = h$. Suppose, however, that the series

$$a_0 h + a_1 \frac{h^2}{2} + a_2 \frac{h^3}{3} + \cdots$$

converges. Show that the former series can be integrated term-by-term in the interval $0 \leqq x \leqq h$.

Formulate precisely each item, or detail, in the theorem you are asked to prove.

*Example:*

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots .$$

7. Consider the series of the Theorem proved in the text. Show that the series

$$\int_a^x u_1(x)\,dx + \int_a^x u_2(x)\,dx + \cdots$$

converges uniformly in the interval $(a, b)$.

8. Does the series for which

$$s_n(x) = \frac{1}{1 + nx}, \qquad 0 \leqq x \leqq 1,$$

converge uniformly? Can it be integrated term-by-term?

Plot the first few approximation curves.

9. The same for

$$s_n(x) = \frac{2n^2x}{1 + n^2x^2}, \qquad 0 \leqq x \leqq 1.$$

10. Show that the series

$$\frac{1}{1^2 + x^2} + \frac{1}{2^2 + x^2} + \frac{1}{3^2 + x^2} + \cdots$$

converges uniformly in the interval $0 \leqq x < \infty$. Can it be integrated from 0 to $\infty$?

11. It can be shown (cf. Chap. VIII) that the function

$$s_n(x) = \frac{\sin x}{1} + \frac{\sin 2x}{2} + \cdots + \frac{\sin nx}{n}$$

is bounded for all values of $n$ and $x$. Show that the series

$$\frac{\sin x}{1} + \frac{\sin 2x}{2} + \cdots$$

can be integrated term-by-term throughout any interval.

12. Prove that the series

$$\sum_{n=1}^{\infty} \frac{2x}{n^2 + x^2}$$

can be integrated term-by-term.

**§ 9. Differentiation of Series.** Let a convergent series of differentiable functions be given:

1) $$f(x) = u_1(x) + u_2(x) + \cdots, \qquad a \leqq x \leqq b.$$

Can it be differentiated term-by-term? i. e. will

2) $$f'(x) = u_1'(x) + u_2'(x) + \cdots$$

be a true equation?

This question, like the corresponding one for integration term-by-term, consists of three parts:—

a) Will the series

3) $$u_1'(x) + u_2'(x) + \cdots$$

converge?

b) Can the function $f(x)$ be differentiated? i. e. does

4) $$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

approach a limit when $\Delta x$ approaches $0$?

c) In case the answers to Questions a) and b) are both affirmative, will 2) be a true eqution?

Consider the approximation curves

5) $$y = s_n(x),$$

where

$$s_n(x) = u_1(x) + \cdots + u_n(x).$$

Each one has a tangent at every point, and so

$$s_n'(x_0)$$

means the slope of the $n$-th approximation curve at the point $x = x_0$.

Consider the series 3). Let

6) $$S_n(x) = u_1'(x) + \cdots + u_n'(x).$$

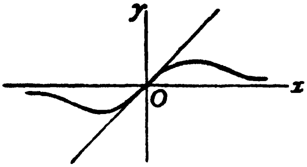
The series 3) converges at a point $x = x_0$ if $S_n(x_0)$ approaches a limit; and conversely. Now

7) $$S_n(x) = s_n'(x).$$

Hence Question a) is the question of whether the slope of the curve 5) at a point $x = x_0$ approaches a limit.

*Example.* Let*

$$S_n(x) = \frac{x}{1+n^2x^2}, \qquad -\infty < x < \infty.$$



The series 1) converges and

$$f(x) = 0.$$

The function $S_n(x)$ is here:

$$S_n(x) = \frac{1-n^2x^2}{(1+n^2x^2)^2}.$$

At the origin,

$$S_n(0) = 1, \qquad \lim_{n=\infty} S_n(0) = 1.$$

Thus the series of the derivatives, 3), converges. The function $f(x)$ has a derivative at the point $x = 0$:

$$f'(0) = 0.$$

But the derivative of the series, namely, $f'(0) = 0$; and the series of the derivatives, namely,

$$1 = u_1'(0) + u_2'(0) + \cdots,$$

are not equal.

A *sufficient* condition for differentiating a series term-by-term is the following.

THEOREM. *Let the terms of the series*

$$u_1(x) + u_2(x) + \cdots \tag{8}$$

*have continuous** derivatives in the closed interval*

$$a \leqq x \leqq b,$$

---

* To plot the curve

$$y = \frac{x}{1+n^2x^2},$$

begin with the curve:

$$y = \frac{x}{1+x^2}.$$

Then make the transformation of similitude,

$$x' = \frac{x}{n}, \qquad y' = \frac{y}{n}.$$

** The restriction of continuity can be removed by means of a different proof, cf. § 10.

*and let the series converge. Let the series of the derivatives,*

9) $$u_1'(x) + u_2'(x) + \cdots,$$

*converge uniformly in the interval. Then the series can be differentiated term-by-term:*

10) $$f(x) = u_1(x) + u_2(x) + \cdots$$

11) $$f'(x) = u_1'(x) + u_2'(x) + \cdots$$

Observe all that the theorem contains. It is first and foremost an *existence theorem* respecting the derivative of $f(x)$. It asserts that, at an arbitrary point $x = x'$ of the interval, the variable

12) $$\frac{f(x' + \Delta x) - f(x')}{\Delta x}$$

approaches a limit as $\Delta x$ approaches 0. Finally, it identifies the derivative $f'(x)$ with the value of the term-by-term derivative series.

*Proof.* Let

13) $$\varphi(x) = u_1'(x) + u_2'(x) + \cdots.$$

The function $\varphi(x)$ is seen to be continuous by § 4. And by § 8 the series 13) can be integrated term-by-term:

14) $$\int_{x_0}^{x} \varphi(x)\,dx = \int_{x_0}^{x} u_1'(x)\,dx + \int_{x_0}^{x} u_2'(x)\,dx + \cdots$$

$$= [\,u_1(x) - u_1(x_0)\,] + [\,u_2(x) - u_2(x_0)\,] + \cdots$$

By hypothesis, the series 8) converges for $x = x_0$:

15) $$f(x_0) = u_1(x_0) + u_2(x_0) + \cdots.$$

On adding 14) and 15) we find:

$$\int_{x_0}^{x} \varphi(x)\,dx + f(x_0) = u_1(x) + u_2(x) + \cdots.$$

Hence

$$f(x) = \int_{x_0}^{x} \varphi(x)\,dx + f(x_0).$$

The function on the right of this equation has a derivative, since

$$\frac{d}{dx}\int\limits_{x_0}^{x} \varphi(x)\,dx = \varphi(x).$$

Therefore $f(x)$ has a derivative, and

$$f'(x) = \varphi(x).$$

This completes the proof.

We could have stated and proved a more general theorem, since we have used the hypothesis of the convergence of 8) only for a single point. If, then, we had demanded that 8) converge in *one* point, $x = x_0$, of the interval, our proof would have shown that 8) necessarily *converges in the whole interval.*

But this formulation of the theorem would be unfortunate for the needs of practice, since in the applications of the theorem one always knows in advance that the series 8) converges, and so this other formulation, by stating and proving what is known before hand, would have distracted attention from the main hypothesis, which one must show is fulfilled, namely, the *uniform convergence of* 9).

It has turned out that the series 8) converges uniformly. But this property would not help in the formulation of a sufficient condition. The series of the above Example converges uniformly, but it cannot be differentiated term-by-term.

*Example.* A power series can be differentiated term-by-term. Let

16) $$a_0 + a_1 x + a_2 x^2 + \cdots$$

converge in the interval

$$-R < x < R.$$

Consider the term-by-term derivative series:

17) $$a_1 + 2a_2 x + 3a_3 x^2 + \cdots.$$

Its terms are continuous and it converges in the above interval; § 5, Exercise 1.

Let $x'$ be any point of the interval. It is possible to find a positive number $h$ such that $|x'| < h < R$. In the interval $(-h, h)$ the series 17) converges uniformly. Hence 16) can be differentiated term-by-term in the point $x'$. But $x'$ is any point of the interval.

Observe the order of choice: — first, $x'$; then the interval $(-h, h)$. It would not be possible to prove the theorem by applying the test to the interval $(-R, R)$, since a power series does not in general converge uniformly in its interval of convergence.

*Remark.* The corresponding theorem in the complex domain is simpler. If $u_n(z)$ is analytic in a two-dimentional region $S$ of the complex $z$-plane, and if the series

$$u_1(z) + u_2(z) + \cdots$$

converges uniformly in $S$, then it defines or represents a function $f(z)$, analytic in $S$, and the series can be differentiated term-by-term in $S$.

### EXERCISES

Show that the following series can be differentiated term-by-term.

1. $$\sum_{n=1}^{\infty} \frac{2x}{n^2 - x^2}.$$

2. $$\sum_{n=1}^{\infty} \frac{1}{(n-x)^2}.$$

3. $$\sum_{n=1}^{\infty} \left[\frac{1}{n-x} - \frac{1}{n}\right].$$

4. $$\sum_{n=1}^{\infty} \left[\log\left(1 + \frac{x}{n}\right) - x \log\left(1 + \frac{1}{n}\right)\right].$$

5. Show that the series

$$\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{a+\pi}} + \frac{1}{\sqrt{a+2\pi}} - \cdots$$

converges uniformly in the interval $0 < a < +\infty$

6. Show that the series of Question 5 can be differentiated term by term.

7. If, in Ex. 3, § 4, $u_n(x)$ has a continuous derivative and the series

$$u_1'(x) + u_2'(x) + \cdots$$

satifies an $M$-test, the function $f(x)$ will have a derivative given by the series

$$\frac{f'(x)}{f(x)}=\sum_{n=1}^{\infty}\frac{u_n'(x)}{1+u_n(x)},$$

provided no factor vanishes.

Complete the theorem for the case that a factor vanishes.

**§ 10. Double Limits and the s (n, m) - Theorem.** The foregoing cases of the continuity of a series, its integration term-by-term, and its differentiation term-by-term, are examples of *double limits*. Thus, for continuity, we start with a convergent series of continuous functions:

$$f(x)=u_1(x)+u_2(x)+\cdots$$

and inquire when $f(x)$ will approach a limit as $x$ approaches $x_0$, that limit to be $f(x_0)$:

1) $$\lim_{x=x_0} f(x)=f(x_0).$$

Now

$$f(x)=\lim_{n=\infty} s_n(x),$$

and so the left hand side of 1) can be written:

2) $$\lim_{x=x_0}\left[\lim_{n=\infty} s_n(x)\right].$$

On the other hand, the right hand side of 1) can be written as

3) $$\lim_{n=\infty}\left[\lim_{x=x_0} s_n(x)\right]$$

For, $s_n(x)$ is continuous at $x=x_0$ and so

$$s_n(x_0)=\lim_{x=x_0} s_n(x);$$

and

$$f(x_0)=\lim_{n=\infty} s_n(x_0).$$

The question of continuity reduces, then, to the question of when the double limits 2) and 3) will be equal, or

4) $$\lim_{x=x_0}\left[\lim_{n=\infty} s_n(x)\right]=\lim_{n=\infty}\left[\lim_{x=x_0} s_n(x)\right].$$

Again, in integration term-by-term, the question is: When will

5) $$\int_a^b \left(\lim_{n=\infty} s_n(x)\right) dx = \lim_{n=\infty} \int_a^b s_n(x)\,dx\,?$$

Since the definite integral is itself by definition a limit, we have here, too, the formulation of the problem in terms of the equality of two double limits.

And, thirdly, in differentiating a series term-by-term. Here,

6a) $$f'(x) = \lim_{\Delta x=0}\left[\lim_{n=\infty} \frac{s_n(x+\Delta x) - s_n(x)}{\Delta x}\right],$$

whereas

6b) $$u_1'(x) + u_2'(x) + \cdots$$

$$= \lim_{n=\infty}\left[\lim_{\Delta x=0} \frac{s_n(x+\Delta x) - s_n(x)}{\Delta x}\right].$$

That two double limits are not in general equal, even when both exist, has appeared time and again in the foregoing paragraphs. But if this is the point to be illustrated, a far simpler example can be given. Consider the function

$$\varphi(x, y) = \frac{2x + 3y}{x + y}, \qquad 0 < x, \quad 0 < y.$$

Here,

$$\lim_{x=0} \varphi(x, y) = 3,$$

and so

$$\lim_{y=0}\left[\lim \varphi(x, y)\right] = 3.$$

But

$$\lim_{y=0} \varphi(x, y) = 2,$$

$$\lim_{x=0}\left[\lim_{y=0} \varphi(x, y)\right] = 2.$$

Thus each of the double limits converges, but their values are not the same.

The short proof given in the older books on the Calculus for the theorem :

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}$$

was based on the assumption that, when each of two double limits exists, their values are equal. And the same criticism holds for applying the rule for determining the limit $\frac{0}{0}$:

$$\lim_{x=a}\frac{f(x)}{F(x)}=\lim_{x=a}\frac{f'(x)}{F'(x)},$$

a second time, when $f'(a)=0\quad F'(a)=0$, thus arriving at the result:

$$\lim_{x=a}\frac{f(x)}{F(x)}=\lim_{x=a}\frac{f''(x)}{F''(x)},\quad \text{etc.}$$

It was not until the middle of the last century that this procedure was justified by sound proofs.

The following theorem has a wide range of applications in questions relating to double limits. It is an outgrowth of the theorem of § 4, Exercise, and it will be convenient to state it first in that restricted form. The notation, $s(m,n)$, is so chosen as to suggest the sum of the first $n$ terms of a series,

$$u_1(m)+u_2(m)+\cdots,$$

whose terms depend on a parameter, $m$:

$$s_n(m)=u_1(m)+\cdots+u_n(m),$$

$$s(m,n)=s_n(m).$$

THEOREM. *Let $s(m,n)$ be a function of the two natural numbers, $m$ and $n$, which satisfies the following conditions:*

a) *$s(m,n)$ approaches a limit when $n$ becomes infinite:*

$$\lim_{n=\infty}s(m,n)=f(m).$$

b) *$s(m,n)$ approaches a limit when $m$ becomes infinite:*

$$\lim_{m=\infty}s(m,n)=S_n.$$

c) *$s(m,n)$ converges uniformly, when $n=\infty$:*

$$|s(m,n')-s(m,n)|<\epsilon,\qquad \nu\leqq n,n',$$

*where $\nu$ does not depend on $m$.*

*Then:*

i) $f(m)$ *approaches a limit, when* $m = \infty$:

$$\lim_{m=\infty} f(m) = A.$$

ii) $S_n$ *approaches a limit, when* $n = \infty$:

$$\lim_{n=\infty} S_n = B.$$

iii) $$A = B.$$

Or:

7) $$\lim_{m=\infty}\left[\lim_{n=\infty} s(m, n)\right] = \lim_{n=\infty}\left[\lim_{m=\infty} s(m, n)\right].$$

*Proof.* In Condition c) let $m = \infty$:

8) $$|S_{n'} - S_n| \leqq \epsilon, \qquad \nu \leqq n, n'.$$

Hence $S_n$ approaches a limit; denote it by $B$:

$$\lim_{n=\infty} S_n = B.$$

In 8) let $n' = \infty$:

$\alpha$) $$|B - S_n| \leqq \epsilon, \qquad \nu \leqq n.$$

In Condition c) let $n' = \infty$:

$\beta$) $$|f(m) - s(m, n)| \leqq \epsilon, \qquad \nu \leqq n.$$

Finally, let $\mu$ be so determined that

$\gamma$) $$|S_\nu - s(m, \nu)| < \epsilon, \qquad \mu \leqq m.$$

On writing $\alpha$) and $\beta$) for $n = \nu$ and combining with $\gamma$) we have:

$$|B - f(m)| < 3\epsilon, \qquad \mu \leqq m.$$

Hence $f(m)$ approaches a limit, $A$; and $A = B$.

Both theorem and proof admit an immediate generalization as follows. Let $\{x\}$ be an arbitrary point set having a cluster point $x = a$, or extending to $+\infty$ $(a = +\infty)$ or to $-\infty$ $(a = -\infty)$. Similarly, let $\{y\}$ be a point set with $y = b$ as cluster point (in particular: $b = +\infty, -\infty$). Let $s(x, y)$ be a function defined for each point $x$ of $\{x\}$ and for each $y$ of $\{y\}$, where $x, y$ are chosen arbitrarily and independently. And now the hypothesis is:

a) $\lim\limits_{y=b} s(x, y)$ shall exist; denote it by $f(x)$.

b) $\lim\limits_{x=a} s(x, y)$ shall exist; denote it by $\varphi(y)$.

c) $| s(x, y') - s(x, y) | < \epsilon$, $\begin{cases} 0 < | y - b | < \delta, \\ 0 < | y' - b | < \delta, \end{cases}$

where $\delta$ is independent of $x$.

The conclusion is:

i) $f(x)$ approaches a limit:

$$\lim_{x=a} f(x) = A.$$

ii) $\varphi(y)$ approaches a limit:

$$\lim_{y=b} \varphi(y) = B.$$

iii) $$A = B.$$

Or:

9) $$\lim_{x=a}\left[\lim_{y=b} s(x, y)\right] = \lim_{y=b}\left[\lim_{x=a} s(x, y)\right].$$

*Remark.* From the hypotheses of the theorem it does not follow that in Condition c) the rôles of $m$ and $n$ can be reversed. It can, indeed, be inferred that, to an arbitrary positive $\epsilon$, correspond two numbers, $p$ and $q$, such that

$$| s(m', n') - s(m, n) | < \epsilon, \qquad p \leqq m, m'; \qquad q \leqq n, n'.$$

If we set $n' = n$, we get:

$$| s(m', n) - s(m, n) | < \epsilon, \qquad p \leqq m, m'; \qquad q \leqq n.$$

But $q$ depends in general on $\epsilon$. As an example let

10) $$s(m, n) = \frac{\sin(n/m)}{n}.$$

Of course, there is nothing paradoxical in this situation. The existence of a two-dimensional limit, as

11) $$\lim_{(m, n) = (\infty, \infty)} s(m, n),$$

does not in general carry with it the existence of either of the one-dimensional limits:

12) $$\lim_{n=\infty} s(m, n), \qquad \lim_{m=\infty} s(m, n).$$

Example:

13) $$s(m, n) = \frac{\sin m}{n} + \frac{\sin n}{m}.$$

And conversely, the existence of both these one-dimensional limits is not enough for the existence of the two-dimensional limit 11): —

*Example:*

$$14)\qquad s(m, n) = \frac{n\,m}{n^2 + m^2}\,.$$

The uniform condition c), combined with the one-dimensional limit $12_2$), does insure the existence of the two-dimensional limit 11).

*A Further Extension.* Let $\{x\}$ denote a point set in space of $n$ dimensions, and let $(a, \cdots, a_n)$ be a cluster point of the points $(x_1, \cdots, x_n)$ of the set. Let $\{y\}$ be a point set in space of $m$ dimensions, and let $(b_1, \cdots, b_m)$ be a cluster point of the points $(y_1, \cdots, y_m)$ of this set. In particular, one or more of the $a$'s may be infinite, $a_k = +\infty$, $-\infty$, or $\infty$; and similarly for the $b$'s.

Let $s(x, y) = s(x_1, \cdots, x_n; y_1, \cdots, y_n)$ be defined for each point $(x)$ of $\{x\}$ and for each $(y)$ of $\{y\}$, where $(x)$ and $(y)$ are chosen arbitrarily and independently.

The last form of statement of the theorem can be interpreted as it stands for the present case. But to avoid so much abbreviation we will say —

*Hypothesis:*

a) $$\lim_{(y)=b)} s(x_1, \cdots, x_n; y_1, \cdots, y_m)$$

*shall exist. Denote it by*

$$f(x_1, \cdots, x_n).$$

b) $$\lim_{(x)=(a)} s(x_1, \cdots, x_n; y_1, \cdots, y_m)$$

*shall exist. Denote it by*

$$\varphi(y_1, \cdots, y_m).$$

c) $$| s(x_1, \cdots, x_n; y_1', \cdots, y_m') - s(x_1, \cdots, x_n; y_1, \cdots, y_m) | < \epsilon,$$

$$0 < | y_k' - b_k | < \delta, \quad 0 < | y_k - b_k | < \delta, \quad k = 1, \cdots, m,$$

*where $\delta$ is independent of $(x_1, \cdots, x_n)$.*

*Conclusion:*

i) *$f(x_1, \cdots, x_n)$ approaches a limit:*

$$\lim_{(x)=(a)} f(x_1, \cdots, x_n) = A.$$

ii) *$\varphi(y_1, \cdots, y_m)$ approaches a limit:*

$$\lim_{(y)=(b)} \varphi(y_1, \cdots, y_m) = B.$$

iii) $$A = B.$$

Or

10) $$\lim_{(x)=(a)} \left[ \lim_{(y)=(b)} s(x_1, \cdots, x_n; y_1, \cdots, y_m) \right] =$$

$$\lim_{(y)=(b)} \left[ \lim_{(x)=(a)} s(x_1, \cdots, x_n; y_1, \cdots, y_m) \right].$$

### EXERCISE

Show that, in the $s(m, n)$-Theorem, Condition c) can be replaced by the following:—

c) *$s(m, n)$ satisfies the relation:*

$$|s(m, n') - s(m, n)| < \epsilon, \qquad \mu \leqq m, \qquad \nu \leqq n, n',$$

*where $\mu$ and $\nu$ both depend on $\epsilon$.*

The conclusion will be the same as before.

**§ 11. Application: Differentiation of Series.** The theorem of § 9 is adequate for the needs of practice. If is of interest, however, to note the following

THEOREM. *Let the series*

1) $$u_1(x) + u_2(x) + \cdots$$

*converge in the interval*

$$a < x < b.$$

*Let $u_n(x)$ have a derivative and let the series*

2) $$u_1'(x) + u_2'(x) + \cdots$$

*converge uniformly in an arbitrary subinterval whose end points lie in the interval. Then the function*

3) $$f(x) = u_1(x) + u_2(x) + \cdots$$

*has a derivative and*

4) $$f'(x) = u_1'(x) + u_2'(x) + \cdots.$$

*Proof.* Let

5) $$s_n(x) = u_1(x) + \cdots + u_n(x).$$

Let $x_0$ and $x_0+\Delta x$ be two points of the interval. Let

$$s(\Delta x, n)=\sum_{k=1}^{n}\frac{u_k(x_0+\Delta x)-u_k(x_0)}{\Delta x}$$

Then i) $$\lim_{\Delta x=0} s(\Delta x, n)=s_n'(x_0);$$

ii) $$\lim_{n=\infty} s(\Delta x, n)=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x},$$

and we wish to show that

$$\lim_{n=\infty}\left[\lim_{\Delta x=0} s(\Delta x, n)\right]=\lim_{\Delta x=0}\left[\lim_{n=\infty} s(\Delta x, n)\right]. \tag{6}$$

This conclusion will be justified if

iii) $$|s(\Delta x, n')-s(\Delta x, n)|<\epsilon, \qquad \nu \leqq n, n',$$

where $\epsilon>0$ is arbitrary and $\nu$ is independent of $\Delta x$. Now:

$$s(\Delta x, n')-s(\Delta x, n)=\sum_{k=n+1}^{n'}\frac{u_k(x_0+\Delta x)-u_k(x_0)}{\Delta x}. \tag{7}$$

Let $$\varphi(x)=s_{n'}(x)-s_n(x).$$

Then the right hand side of 7) has the value

$$\frac{\varphi(x_0+\Delta x)-\varphi(x_0)}{\Delta x}=\varphi'(x_0+\theta\Delta x)=$$

$$s_{n'}'(x_0+\theta\Delta x)-s_n'(x_0+\theta\Delta x).$$

Because of the uniform convergence of the series 2) we have:

$$|s_{n'}'(x)-s_n'(x)|<\epsilon, \qquad \nu \leqq n, n',$$

where $\nu$ is independent of $x$, no matter where $x$ lies, and hence when, in particular, $x=x_0+\theta\Delta x$. This completes the proof.

The theorem is more general than the test of § 9 in that it does not presuppose the continuity of the derivatives $u_n'(x)$. It is less general in that it demands the convergence of the series 1) for every $x$ instead of for a single $x$. The theorem is found in Harnack's *Differential- und Integralrechnung*, § 129.

**§ 12. Condensation of Singularities.** In the examples of non-uniformly convergent series hitherto considered, the removal of the neighborhood of a single point of the interval of definition of the terms yields a new interval, in which the series converges uniformly. This situation is not characteristic for the general case. A series of continuous functions may converge toward a continuous function:

$$f(x) = u_1(x) + u_2(x) + \cdots, \qquad a \leqq x \leqq b,$$

and yet the convergence may be non-uniform in every subinterval:

$$a \leqq a' \leqq x \leqq b' \leqq b.$$

If is easy to construct such examples by a *Method of Condensation of Singularities*, due to Hankel. Such an example is studied in the Author's *Funktionentheorie*, vol. I, 1928, p. 92, and illusstrated by graphs. Starting with the function

$$1) \qquad y = \psi(x) = \sqrt{2e}\, x e^{-x^2},$$

the graph of which is readily plotted, form the function

$$2) \qquad \varphi_n(x) = \psi(n \sin^2 \pi x).$$

It is now easy to plot the graphs of the functions:

$$y = \varphi_n(x), \quad y = \frac{1}{2!}\varphi_n(2!x), \quad y = \frac{1}{3!}\varphi_n(3!x), \cdots.$$

Form the series, the sum of whose first $n$ terms is:

$$3) \qquad s_n(x) = \varphi_n(x) + \frac{1}{2!}\varphi_n(2!x) + \cdots + \frac{1}{n!}\varphi_n(n!x).$$

*This series converges to the value* 0 *for every* $x$, *but it converges non-uniformly in every interval.*

A further important application of this Principle is to the formation of non-analytic functions of real variables. A function of the real variable, $x$, is said to be *analytic* at a point, $x = x_0$, if it can be developed into a power series in the neighborhood of the point, i. e. developed by Taylor's Theorem. The classical example, due to Cauchy, of a function which is continuous, together with its derivatives of all orders, and yet cannot be developed by Taylor's Theorem, is the following:

$$4) \qquad \begin{cases} f(x) = e^{-\frac{1}{x^2}} & x \neq 0; \\ f(0) = 0. \end{cases}$$

For the point $x_0 = 0$ all the derivatives vanish. The Taylor's expansion:

$$f(x_0) + x f'(x_0) + \frac{x^2}{2!} f''(x_0) + \cdots$$

converges, but it represents, not the above function, $f(x)$, but the function 0.

Cauchy's example has this pecularity in only one point. It is easy, by Hankel's Principle, to construct examples of functions which have derivatives of all orders in every point, but which cannot be developed by Taylor's Theorem in any interval. Such an example is the following. Let

5) $$\varphi(x) = f(\sin \pi x),$$

where $f(x)$ is the function 4). Then

5) $$\Phi(x) = \sum_{n=1}^{\infty} \frac{\varphi(n!\, x)}{(n!)^n}$$

is the desired function; l. c. p. 126.

## EXERCISES ON CHAPTER V.

1. Show that the series:

$$\sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta),$$

whose coefficients $a_n, b_n$ are bounded, represents a function which is continuous, together with all its derivatives, within the circle $r < 1$, and statisfies Laplace's Equation·

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

2. Let

$$c_1 + c_2 + \cdots$$

be a convergent series. Show that the series:

$$c_1 \frac{\operatorname{sh} t}{\operatorname{sh} a} + c_2 \frac{\operatorname{sh} 2t}{\operatorname{sh} 2a} + \cdots$$

converges uniformly in the interval $0 \leqq t \leqq a$.

3. Does the series

$$\sum_{n=1}^{\infty} \frac{2x}{n^2 - x^2}$$

converge uniformly:

i) in the interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$?

ii) in the interval $-1 < x < 1$?

iii) in the whole interval $-\infty < x < \infty$, the points $x = 0, \pm 1, \pm 2, \cdots$ having been removed?

4. Let the series

$$u_1(x) + u_2(x) + \cdots$$

converge uniformly in the interval

$$a \leqq x < b$$

and let it converge in the point $x = b$. Show that it converges uniformly in the closed interval

$$a \leqq x \leqq b.$$

5. Let

$$f(x) = u_1(x) + u_2(x) + \cdots$$

be a series which converges uniformly in the interval

$$a \leqq x < b,$$

the terms being continuous in the closed interval $a \leqq x \leqq b$. Show that $f(x)$ approaches a limit when $x$ approaches $b$.

6. Let $\varphi_n(x)$ be defined in the interval $a < x < b$, and let the sum

$$\varphi_1(x) + \varphi_2(x) + \cdots + \varphi_n(x),$$

regarded as a function of $x$ and $n$, be bounded. Let

$$\alpha_1 \geqq \alpha_2 \geqq \cdots, \qquad \lim_{n=} \alpha_n = 0,$$

be a set of constants. Show that the series

$$\alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x) + \cdots$$

converges uniformly.

7. Let $\varphi_1(x), \varphi_2(x), \cdots$ be defined in the interval $a < x < b$ and let

$$\varphi_{n'}(x) \leqq \varphi_n(x), \quad n < n';$$

$$\lim_{n=\infty} \varphi_n(x) = 0.$$

Let $c_1, c_2, \cdots$ be a set of constants such that

$$c_1 + c_2 + \cdots + c_n$$

is bounded, $n = 1, 2, \cdots$. Show that the series

$$c_1 \varphi_1(x) + c_2 \varphi_2(x) + \cdots$$

converges.

Give an example of such a series which does not converge uniformly.

State a sufficient condition, that the series converge uniformly.

8. Let $f(x, \alpha)$ be a function which, for each $\alpha$ of an infinite point set $A$, is defined for every value of $x$ in the interval

$$a \leqq x \leqq b.$$

Let $\alpha_0$ be a cluster point of $A$, not belonging to $A$. Let $f(x, \alpha)$ approach a limit when $(x, \alpha)$ approaches $(x_0, \alpha_0)$, where $x_0$ is an arbitrary point of the interval $(a, b)$, and $\alpha$ is restricted to the point set $A$. Denote the limit by $\varphi(x_0)$:

$$\varphi(x_0) = \lim_{(x, \alpha)=(x_0, \alpha_0)} f(x, \alpha).$$

Show that $\varphi(x)$ is continuous in the interval $(a, b)$.

9. Prove, furthermore, that the function $f(x, \alpha)$ of the preceding question converges uniformly as $\alpha$ approaches $\alpha_0$.

Is the converse proposition true?—namely: If $f(x, \alpha)$ is defined in the points $a \leqq x \leqq b$, $\alpha$ in $A$, and if $f(x, \alpha)$ converges uniformly as $\alpha$ approaches $\alpha_0$, then $f(x, \alpha)$ approaches a limit when $(x, \alpha)$ approaches $(x_0, \alpha_0)$, where $x_0$ is an arbitrary point of the interval $(a, b)$ and $\alpha$ is restricted to the points of $A$.

10. Can a function which is everywhere discontinuous approach a limit i) non-uniformly; ii) uniformly?

# Chapter VI

## The Elementary Functions

**§1. The Trigonometric Functions.** The noblest branch of Physics is Geometry, or the physical science of space. Next in importance, in the physical sciences, is Kinematics, or the science of motion. The most important class of motions is that of oscillations — the oscillation of a point, the vibration of a membrane, a wave in a three-dimensional region of space. The simplest case is that of Simple Harmonic Motion, dominated by the differential equation:

$$\frac{d^2x}{dt^2} + n^2x = 0.$$

If we change the variable from $t$ to $r$, where

$$r = n\,t,$$

the new equation becomes:

$$\frac{d^2x}{dr^2} + x = 0.$$

It is this differential equation which dominates the whole class of phenomena known as waves, and so it is natural to enquire what the functions are which constitute its solution. We will change the notation and write the differential equation in the form:

$$\text{(A)} \qquad \frac{d^2y}{dx^2} + y = 0.$$

By a *solution* of this equation in a given interval is meant a function,

$$y = f(x),$$

which has a second derivative at each point of the interval and satisfies the differential equation; i.e. causes the left-hand member to vanish identically:

$$f''(x) + f(x) \equiv 0.$$

It is clear that such a function admits continuous derivatives of all orders. For, $f'(x)$ exists by hypothesis, and is continuous, since $f''(x)$ exists. Moreover, $f''(x)$ is continuous, since

$$f''(x) = -f(x).$$

Next, $f'''(x)$ exists and is continuous, since this last equation has on the right-hand side a function with a continuous derivative. And so on.

*Solution by a Power Series.* As the interval of definition we will begin with one including the point $x = 0$, and enquire whether there be a solution given by a power series,

$$y = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Suppose this is the case. Then

$$\frac{d^2 y}{dx^2} = 2\, a_2 + 3 \cdot 2\, a_3 x + \cdots + n(n-1)\, a_n x^{n-2} + \cdots .$$

A necessary condition is obtained by adding these two series. The result is a power series that vanishes identically. Consequently each coefficient must vanish.

$$\begin{array}{ll} a_0 + 2 \cdot 1 a_2 = 0, & a_1 + 3 \cdot 2 a_3 = 0, \\ a_2 + 4 \cdot 3 a_4 = 0, & a_3 + 5 \cdot 4 a_5 = 0, \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

The coefficients $a_0$ and $a_1$ can be chosen arbitrarily. The remaining coefficients are then determined uniquely. Thus, if $a_0 = 0$, $a_1 = 1$, we find the series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots ;$$

and if $a_0 = 1$, $a_1 = 0$, the series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots .$$

Both of there series converge for all values of $x$ and so define two functions :

$$1) \qquad \begin{cases} s(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots \\[2ex] c(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots \end{cases}$$

Conversely, these functions are solutions. For each of them, when substituted in the differential equation (A), is seen to satisfy it. Or it would be possible to start with the equations for the coefficients and show that each step can be retraced.

The function $s(x)$ is odd, and $c(x)$ is even:

$$2)\qquad s(-x) = -s(x), \qquad c(-x) = c(x).$$

Moreover,

$$s(0) = 0, \qquad c(0) = 1.$$

*The Derivatives of* $s(x)$, $c(x)$. Observe that

$$3)\qquad s'(x) = c(x), \qquad c'(x) = -s(x).$$

Hence

$$4)\qquad \begin{cases} s'(x) = \phantom{-}c(x), \\ s''(x) = -s(x), \\ s'''(x) = -c(x), \\ s^{iv}(x) = \phantom{-}s(x), \end{cases}$$

and now the further derivatives repeat themselves periodically in blocks of four.

*The Addition Theorem.* From this property and from the continuity of the functions $s(x)$, $c(x)$ we now infer by the aid of Taylor's Theorem with the Remainder that $s(x)$ can be developed about an arbitrary point, $x = x_0$:

$$5)\qquad s(x_0 + h) = s(x_0) + c(x_0)h - s(x_0)\frac{h^2}{2!} - c(x_0)\frac{h^3}{3!} + \cdots$$

The right-hand side can be written in the form:

$$s(x_0)\left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \cdots\right)$$

$$+\, c(x_0)\left(h - \frac{h^3}{3!} + \frac{h^5}{5!} - \cdots\right).$$

Hence we see that

$$6)\qquad s(x_0 + h) = s(x_0)c(h) + c(x_0)s(h).$$

A similar relation holds for $c(x_0 + h)$. Since $x_0$ and $h$ are both arbitrary, we may write more symmetrically:

$$7)\qquad \begin{cases} s(u + v) = s(u)c(v) + c(u)s(v). \\ c(u + v) = c(u)c(v) - s(u)s(v). \end{cases}$$

Thus we have obtained the Addition Theorem for these two functions.

*The Pythagorean Identity.* Differentiate the function:

$$s(x)^2 + c(x)^2:$$

$$\frac{d}{dx}\left[s(x)^2 + c(x)^2\right] = 2\left[s(x)c(x) - c(x)s(x)\right] = 0.$$

Hence

$$s(x)^2 + c(x)^2 = k.$$

Setting $x = 0$, we find: $k = 1$. Hence

8) $$s(x)^2 + c(x)^2 = 1.$$

We shall see a little later that it is reasonable to describe this relation as the Pythagorean Identity.

*Roots of* $s(x)$, $c(x)$. The function $c(x)$ has a positive root. For, suppose this were not so. Turn to the function $s(x)$. Since $s(0) = 0$, and, by hypothesis,

$$s'(x) = c(x) > 0,$$

it follows that $s(x)$ is positive and increasing for all positive values of $x$. By the Law of the Mean

$$c(x) = c(a) - (x - a)s(X), \qquad a < X < x.$$

If $a > 0$,

$$0 < s(a) < s(X),$$

hence

$$c(x) < c(a) - (x - a)\, s(a).$$

But the right-hand member is negative for large values of $x$, since $s(a) > 0$. From this contradiction follows that $c(x)$ has a positive root. Let $p/2$ be the smallest positive root of $c(x)$.

*Further Identities. Periodicity.* From the Addition Theorem it follows that

9) $$s\left(x + \frac{p}{2}\right) = c(x), \qquad c\left(x + \frac{p}{2}\right) = -s(x).$$

Furthermore,

10) $$s(x + p) = -s(x), \qquad c(x + p) = -c(x).$$

and finally:

11) $$s(x + 2p) = s(x), \qquad c(x + 2p) = c(x).$$

This last identity shows that the functions $s(x)$, $c(x)$ admit the period $2p$. It remains to show that this is a primitive period.

The function $c(x)$ is even, and hence $-p/2$ is the first negative root. The graph of the function

$$y = c(x)$$

in the interval $-p/2 \leqq x \leqq p/2$ is, then, as shown. From this graph we obtain the graph of

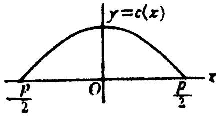
$$y = s(x), \qquad 0 \leqq x \leqq p,$$

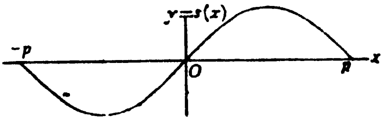
by the transformation

$$x' = x + \frac{p}{2}$$

and the relations

$$c\left(x' - \frac{p}{2}\right) = -c\left(x' + \frac{p}{2}\right) = s(x').$$





Since $s(x)$ is an odd function, the graph in the interval:

$$-p/2 \leqq x \leqq 0$$

is found by rotating this arch about the origin through 180°. Finally, the periodicity gives the complete graph.

That $2p$ is a primitive period is now easily shown. For, let $\omega$ be any positive period. Then

$$s(x + \omega) = s(x).$$

Hence, in particular,

$$s(\omega) = s(0) = 0.$$

Thus $\omega$ must be a multiple of $p$. But the odd multiples are not periods. The figures for $s(x)$ and $c(x)$ are found on p. 196.

*The Simultaneous Equations:* $s(x) = \alpha$, $c(x) = \beta$. Let $h$ have any value between $-1$ and $1$, inclusive:

$$-1 \leqq h \leqq 1.$$

The equation

12) $$c(x) = h$$

has two distinct roots in the interval $-p \leqq x \leqq p$, if $-1 < h < 1$, and these are equal and opposite, $x_1$ and $x_2 = -x_1$. If $h = 1$, there is a single root, $x = 0$; and if $h = -1$, the roots are $x = p$ and $-p$. Only one of these latter, however, should be counted, because of the periodicity. — Similarly for $s(x)$.

If $\alpha$, $\beta$ are any two numbers such that

$$\alpha^2 + \beta^2 = 1, \tag{13}$$

the equations

$$s(x) = \alpha, \qquad c(x) = \beta \tag{14}$$

admit one and only one solution in the interval

$$0 \leqq x < 2p$$

or, generally,

$$a \leqq x < a + 2p,$$

where $a$ has any value whatever. For, consider the second equation in the interval

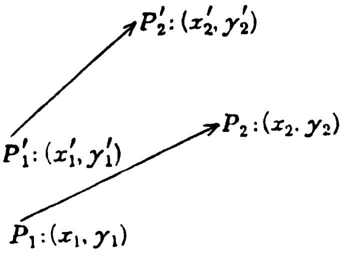
$$-p < x \leqq p.$$

It admits two distinct roots, $x_1$ and $x_2$, where

$$x_2 = -x_1,$$

if $-1 < \beta < 1$. But $s(x_2) = -s(x_1) \neq 0$, and so only one of these satisfies the first equation. — The cases $\beta = 1, -1$ are dealt with directly. They do not form an exception.

*Definition of Angle.* Consider two directed line segments in the plane, $P_1 P_2$ and $P_1' P_2'$. By the *angle*, $\theta$, from $P_1 P_2$ to $P_1' P_2'$ shall be meant any solution of the simultaneous equations:



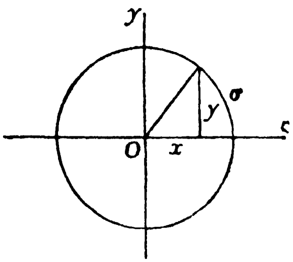
$$c(\theta) = \kappa^{-1} \begin{vmatrix} y_2' - y_1' & -(x_2' - x_1') \\ x_2 - x_1 & y_2 - y_1 \end{vmatrix}$$

15)

$$s(\theta) = \kappa^{-1} \begin{vmatrix} y_2' - y_1' & x_2' - x_1' \\ y_2 - y_1 & x_2 - x_1 \end{vmatrix}$$

$$\kappa = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}\ \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The definition is invariant of the transformations of the principal group (*Hauptgruppe*) i.e. translations, rotations, and stretchings; but a reflection carries $\theta$ over into $-\theta$. Moreover, $\theta$ is invariant of the choice of $P_1$, $P_2$ on a given line, provided merely that the sense of the vector $P_1 P_2$ is preserved; and similarly for $P_1'$, $P_2'$. Finally, the vectors $P_1 P_2$ and $P_1' P_2'$ can be replaced by any equal vectors. — The proofs of all these statements are simple, and are left to the reader.



In particular, then, the lines $P_1 P_2$ and $P_1' P_2'$ can always be replaced by two radii of the unit circle; the first, drawn to the point $(1, 0)$, the second, to the point $(x, y)$. The definition then gives:

16) $$c(\theta) = x, \qquad s(\theta) = y.$$

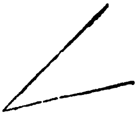
Thus a unique value of $0 \leqq \theta < 2p$ is obtained.

*Equality of Angle and Arc.* The number $\theta$ thus defined is equal to the length $\sigma$ of the arc of the unit circle, measured from $(1, 0)$ in the sense corresponding to the increasing ordinate near this point. For,

$$d\sigma^2 = dx^2 + dy^2 = [c'(\theta)^2 + s'(\theta)^2]\,d\theta^2 = d\theta^2,$$

17) $$d\sigma = d\theta.$$

*Angles in Geometry.* In Elementary Geometry an *angle* is defined as the *figure* made by two lines which have an extremity in common. Each "line" may be a line segment or a ray.



To such an angle is assigned a number — its measure — by choosing an arbitrary angle as the unit and applying it to the given angle in the usual way. It is this number which we have defined above as the angle $\theta$, with the usual extension of the definition to positive and negative angles.

The functions $s(\theta)$ and $c(\theta)$ are now seen to be identical with the sine and cosine as ordinarily defined in Trigonometry:

18) $$s(\theta) = \sin\theta, \qquad c(\theta) = \cos\theta.$$

*The General Solution of Equation (A).* Let

$$y = f(x)$$

be any solution of Equation $(A)$ in the neighborhood of the point $x = a$. Then two constants, $c_1$ and $c_2$, can be so determined that the solution

$$\varphi(x) = c_1 \sin x + c_2 \cos x,$$

together with its first derivative, will tally with $f(x)$ at $x = a$:

$$f(a) = c_1 \sin a + c_2 \cos a$$

$$f'(a) = c_1 \cos a - c_2 \sin a$$

since the value of the determinant is $-1$. Thus the solution

$$\psi(x) = f(x) - \varphi(x)$$

is such that

$$\psi(a) = 0, \qquad \psi'(a) = 0.$$

Hence $\psi^{(n)}(a) = 0$ for all values of $n$.

Now, any solution of $(A)$:

$$y = F(x),$$

can be developed by Taylor's Theorem about an arbitrary point $x = c$. For,

$$F^{(n)}(x) = \pm F(x) \quad \text{or} \quad \pm F'(x),$$

according as $n$ is even or odd, and so Taylor's Theorem with the Remainder applies. Hence $\psi(x) \equiv 0$, and the theorem is proved.

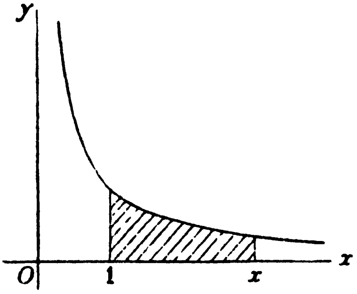
§2. **The Logarithmic Function.** The definition of the logarithm which leads most easily to the properties of this function, is by means of the integral. Let

$$1) \qquad L(x) = \int_1^x \frac{dx}{x}, \qquad 0 < x < \infty.$$

This function will, or course, turn out to be the natural logarithm of $x$. But we are not assuming any knowledge of special functions, except what has already been shown regarding the rational functions — in particular, their continuity — and so it is better to use a notation that suggests $\log x$ without the danger of taking for granted properties not yet established.

From the definition follows at once that

2) $L(1) = 0$; $L(a) < 0,\ 0 < a < 1$; $L(a) > 0,\ a > 1$.



THEOREM 1. *The function $L(x)$ is continuous for all positive values of $x$. It has a derivative, given by the equation:*

3) $$L'(x) = \frac{1}{x}.$$

THEOREM 2. *The function is monotonic increasing:*

4) $$L(x) < L(y), \qquad x < y.$$

For, by the Law of the Mean,

$$L(y) = L(x) + (y - x)\ L'[x + \theta(y - x)],$$

and $L'(x)$ is positive for all values of the argument.

COROLLARY. *From*

$$L(x) = L(y)$$

*follows that*

$$x = y.$$

THEOREM 3. THE FUNCTIONAL RELATION. *The function $L(x)$ satisfies the functional equation:*

(*A*) $$L(x) + L(y) = L(xy)$$

*for all possible values of the arguments.*

Let the left-hand side of Equation (*A*) be written in the form:

$$\int_1^x \frac{dt}{t} + \int_1^y \frac{dr}{r}.$$

In the second integral, change the variable of integration:

$$t = x\,r.$$

Thus

$$\int_1^y \frac{dr}{r} = \int_x^{xy} \frac{dt}{t}.$$

Hence we see that

$$\int_1^x \frac{dt}{t} + \int_1^y \frac{dr}{r} = \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} = \int_1^{xy} \frac{dt}{t}.$$

This last integral has the value:

$$\int_1^{xy} \frac{dt}{t} = L(xy)$$

and the theorem is proved.

By setting $y = 1/x$ in $(A)$ we find:

$$L(x) + L\left(\frac{1}{x}\right) = L(1) = 0.$$

or

5) $$L\left(\frac{1}{x}\right) = -L(x).$$

By setting $y = x$ in $(A)$ we find:

$$L(x^2) = 2\,L(x).$$

Similarly,

6) $$L(x^n) = n\,L(x),$$

where $n$ is any natural number.

THEOREM 4. *The function $L(x)$ becomes positively infinite for $x = +\infty$, and negatively infinite for $x = 0^+$:*

$$L(+\infty) = +\infty; \qquad L(0^+) = -\infty.$$

Set $x = 2$ in 6). Then

$$L(2^n) = n\,L(2).$$

Now, $2^n$ becomes infinite with $n$, since $2^n > n$. The right-hand side becomes infinite with $n$, since $L(2) > 0$. The function $L(x)$ is monotonic increasing. This proves the theorem for $x = +\infty$. The proof of the second part follows from 5).

**§3. The Exponential Function.** The function $L(x)$ admits an inverse, defined and continuous for all values of the argument. Denote it by $E(x)$:

1) $$y = E(x) \qquad \text{if} \qquad x = L(y).$$

It is clear that

2) $$E(x) > 0, \qquad -\infty < x < \infty.$$

Moreover,

3) $$E(-\infty) = 0, \qquad E(0) = 1, \qquad E(+\infty) = +\infty.$$

THEOREM 1. ADDITION THEOREM. *The function $E(x)$ admits the Addition Theorem:*

$(B)$ $$E(x+y) = E(x)\,E(y).$$

This relation is the precise counterpart of $(A)$, §2. For, let $x$ and $y$ be any two real numbers. Then the equations:

$$u = E(x), \qquad v = E(y),$$

admit unique solutions, for $u$ and $v$ are both positive:

$$x = L(u), \qquad y = L(v).$$

Now, by §2, Theorem 3:

$$L(u) + L(v) = L(uv).$$

Hence

$$x + y = L(uv),$$

and so, by definition:

$$E(x+y) = uv,$$

or

$$E(x+y) = E(x)\,E(y), \qquad q.e.d.$$

THEOREM 2. *The function $E(x)$ admits a derivative,*

4) $$E'(x) = E(x).$$

For, if

$$y = E(x), \qquad \text{then} \qquad x = L(y);$$

and if

$$y + \Delta y = E(x + \Delta x), \qquad \text{then} \qquad x + \Delta x = L(y + \Delta y).$$

Thus

$$\Delta x = L(y + \Delta y) - L(y) = \Delta y \; L'(y + \theta y),$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{L'(y + \theta \Delta y)}.$$

Now, $E(x)$ is continuous, (Chap. III, § 9, Ex. 8) and so $\Delta y$ approaches 0 as $\Delta x$ approaches 0. Hence the right-hand side of this last equation approaches a limit as $\Delta x$ approaches 0:

$$\lim_{\Delta x = 0} \frac{1}{L'(y + \theta \Delta y)} = \frac{1}{L'(y)}.$$

But

$$L'(y) = \frac{1}{y}.$$

Hence

$$D_x y = y,$$

or

$$E'(x) = E(x), \qquad q.e.d.$$

THEOREM 3. *The function $E(x)$ can be expanded by Taylor's Theorem for all values of the argument:*

$$5) \qquad E(x_0 + h) = E(x_0) + h E'(x_0) + \frac{h^2}{2!} E''(x_0) + \cdots$$

The proof is given in the Calculus, and need not be repeated here.

In particular, let $x_0 = 0$, $h = x$. Because of 3) and 4),

$$6) \qquad E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Thus 5) turns out to be nothing more or less than the Addition Theorem:

$$E(x_0 + h) = E(x_0) E(h).$$

Observe that we have here an independent proof of the Addition Theorem.

*The Function $a^x$.* We have, incidentally, a new proof for the existence of roots, $\sqrt[q]{a}$, where $a > 0$ is any number. By definition,

$$b = \sqrt[q]{a} \qquad \text{if} \qquad b^q = a.$$

In Equation 6) of § 2 set $n = q$, $x = b$. Then

$$L(b^q) = q L(b),$$

$$L(b)=\frac{1}{q}L(a),$$

7) $$a^{\frac{1}{q}}=E\left[\frac{1}{q}L(a)\right].$$

Equation 6) of § 2 is now seen to hold for all rational values of $n$:

8) $$L\left(a^{\frac{p}{q}}\right)=\frac{p}{q}L(a), \qquad 0<a.$$

Hence

9) $$a^{\frac{p}{q}}=E\left[\frac{p}{q}L(a)\right].$$

Now, the function

$$E[x\,L(a)]$$

is continuous for all values of $x$, and so we can define the function $a^x$ by the equation:

10) $$a^x=E[x\,L(a)].$$

From this equation follows that

11) $$L(a^x)=x\,L(a),$$

or Equation 6) of § 2 holds for $0<x<\infty$, $-\infty<n<\infty$.

*The Natural Base, e.* The functional equation $(B)$ is precisely the one studied in Chap. V, § 7. Hence it appears that, if we define the number $e$ as $E(1)$:

12) $$e=E(1),$$

the function

13) $$E(x)=e^x.$$

The number $e$ can be computed from the series 6):

14) $$e=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

or

$$e=2.\ 71828\ 18284\ 59045\ \cdots$$

The function $E(x)$ having thus been identified with $e^x$, its inverse, $L(x)$, is seen to be $\log x$ by the usual definition of the latter function:

12) $$L(x)=\log x.$$

**§ 4. A Simpler Analytic Treatment.** The foregoing treatment of the Elementary Functions has the advantage that it links the trigonometric functions with the simplest of oscillatory motions, thus emphasizing their periodicity at the outset; and it yields the properties of the logarithm with a minimum amount of computation.

From the standpoint of pure analysis, however, one must admit that there is no simpler limiting process than that of power series. It is possible to make the differential equation

1) $$\frac{dy}{dx} = y$$

the point of departure and to seek a solution in the form of a power series. We arrive immediately at a particular solution:

2) $$E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The derivative of this function has the value, from Equation 1):

3) $$E'(x) = E(x).$$

Now follows the developability of this function by Taylor's Theorem:

4) $$E(x_0 + h) = E(x_0) + h\,E'(x_0) + \frac{h^2}{2!}E''(x_0) + \cdots$$

Hence

$$E(x_0 + h) = E(x_0)\;E(h),$$

and herewith the Addition Theorem:

5) $$E(x + y) = E(x)\;E(y),$$

The theory of this functional equation leads at once to the identification of $E(x)$ with $e^x$:

6) $$E(x) = e^x.$$

In particular, then:

7) $$E(x) > 0, \qquad -\infty < x < \infty;$$

8) $$E(-\infty) = 0^+, \qquad E(0) = 1, \qquad E(+\infty) = +\infty.$$

The existence and the properties of the function $a^x$, and the treatment of the logarithm as the inverse function, follow without artificialities of any sort.

*The Trigonometric Functions.* The use of imaginaries is so familiar to the electrical engineer of the present day, that the $\sqrt{-1}$ can no longer be regarded as a mathematical fiction, even by the practical man. If, then, starting with the traditional equation

$$9)\qquad e^{\varphi i} = \cos\varphi + i\sin\varphi, \qquad i = \sqrt{-1},$$

we define two new functions by the equations:

$$\begin{cases} S(x) = \dfrac{e^{xi} - e^{-xi}}{2i}, \\[2ex] C(x) = \dfrac{e^{xi} + e^{-xi}}{2}, \end{cases}$$

we prove immediately by elementary algebra that these functions have the properties:

$$11)\qquad \begin{cases} S(x+y) = S(x)\,C(y) + C(x)\,S(y); \\ C(x+y) = C(x)\,C(y) - S(x)\,S(y). \end{cases}$$

$$12)\qquad S'(x) = C(x), \quad C'(x) = -S(x).$$

$$13)\qquad S(x)^2 + C(x)^2 = 1.$$

$$14)\qquad \begin{cases} S(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots \\[2ex] C(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots \end{cases}$$

Finally, the periodicity is established as in § 1, and by means of the angle these functions become identified with $\sin x$, $\cos x$:

$$15)\qquad S(x) = \sin x, \qquad C(x) = \cos x.$$

**§ 5. Partial Fractions. Development of cot x.** The functions $\tan x$, $\cot x$, $\sec x$, $\csc x$ can be represented by infinite series which are analogous to the representation of a fraction by partial fractions. The fundamental equation is the following:

$$(A)\qquad \pi\cot\pi x = \frac{1}{x} - \sum_{n=1}^{\infty}\frac{2x}{n^2 - x^2}.$$

This equation can be obtained immediately from the development of a function into a Fourier's series; cf. Chap. VIII, § 1. An elementary deduction can be given by means of de Moivre's Theorem:

$$\cos m\varphi = \cos^m \varphi - \frac{m(m-1)}{1\cdot 2} \cos^{m-2}\varphi \sin^2\varphi +$$

$$\frac{m(m-1)(m-2)(m-3)}{4!} \cos^{m-4}\varphi \sin^4\varphi - \cdots$$

$$\sin m\varphi = m\cos^{m-1}\varphi \sin\varphi - \frac{m(m-1)(m-2)}{3!} \cos^{m-3}\varphi \sin^3\varphi$$

$$+ \cdots$$

We see that $\cot m\varphi$ is a rational function of $\tan\varphi$:

$$\cot m\varphi = \frac{g(\tan\varphi)}{G(\tan\varphi)}.$$

Let $m$ be odd: $m = 2\mu + 1$. The function on the right is a proper fraction, as is seen by allowing $\varphi$ to approach the limit $\pi/2$; for then the left hand side approaches 0 as $\tan\varphi$ becomes infinite.

We proceed to represent this fraction by means of partial fractions. The degree of the denominator is obviously not greater than $m$. On the other hand we can write down $m$ distinct values of $-\pi/2 < \varphi < \pi/2$, for which $\cot m\varphi$ becomes infinite, namely:

$$\varphi = 0, \quad \pm\frac{\pi}{m}, \quad \pm\frac{2\pi}{m}, \quad \cdots, \quad \pm\frac{\mu\pi}{m}.$$

Consequently,

$$G(\tan\varphi) = \tan\varphi \left(\tan\varphi - \tan\frac{\pi}{m}\right)\left(\tan\varphi + \tan\frac{\pi}{m}\right)\cdots$$

We thus find:

$$\cot m\varphi = \sum_{k=-\mu}^{\mu} \frac{A_k}{\tan\varphi - \tan\dfrac{k\pi}{m}}.$$

To determine the coefficients $A_k$, multiply through by $\sin m\varphi$ and then let $\varphi$ approach the limit $k\pi/m$. All the terms on the right approach 0 except the term in $A_k$, which takes on the indeterminate form 0/0. The limiting value is found by the usual method of the calculus. Thus

$$\cos k\pi = \frac{A_k m \cos k\pi}{\sec^2 \frac{k\pi}{m}}, \qquad A_k = \frac{1}{m} \sec^2 \frac{k\pi}{m}.$$

On setting $m\varphi = x$ we obtain the final formula:

$$1) \qquad \cot x = \sum_{k=-\mu}^{\mu} \frac{\sec^2 \frac{k\pi}{m}}{m \tan \frac{x}{m} - m \tan \frac{k\pi}{m}},$$

or:

$$2) \qquad \cot x = \frac{1}{m \tan \frac{x}{m}} + \sum_{k=1}^{\mu} \frac{2 \sec^2 \frac{k\pi}{m}\, m \tan \frac{x}{m}}{m^2 \tan^2 \frac{x}{m} - m^2 \tan^2 \frac{k\pi}{m}}.$$

Allow $m$ to become infinite. The terms on the right of 2) approach limits, and it would not have occurred to the mathematicians of even a hundred years ago to question the inference that

$$3) \qquad \text{ot}\ x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2 \pi^2}.$$

This result is correct, but it requires proof.

The right hand side of 2) can be regarded as an infinite series whose terms depend on $m$ ($x$ is a constant, $\neq \pm k\pi$) and which converges for* $m = 1, 3, 5, \cdots$:

$$f(m) = u_0(m) + u_1(m) + \cdots$$

The terms of this series approach limits as $m = \infty$; namely, the corresponding terms of 3). And now the series of the limits, i.e. the right-hand side of 3). and the limit of the series, i.e. the left-hand side of 3), will be equal if the series converges uniformly; cf. Chap. V, § 4, Ex. 1. or § 10. That this is the case can be shown by an $M$-test. We have already chosen $x$. Let $A > |x|$. Then

$$\left| m \tan \frac{x}{m} \right| < A, \qquad m_0 \leqq m.$$

---

* It may happen that for small values of $m$ the function $\tan \frac{x}{m}$ is not defined. In that case, begin with a larger $m$; i. e. let $\nu \leqq m$.

On the other hand, the function $f(x)$:

$$\begin{cases} f(x) = \dfrac{\sin x}{x}, & 0 < x \leqq \dfrac{\pi}{2} \\ f(0) = 1 \end{cases}$$

is continuous and positive in the closed interval $(0, \pi/2)$, and hence its minimum value, $h$, is positive. Hence

$$0 < h < \frac{\sin x}{x}, \qquad 0 < x \leqq \frac{\pi}{2}.$$

If, then, we set

$$M_k = \frac{2A}{k^2\pi^2 h^2 - A^2}, \qquad k_0 \leqq k,$$

we have here an $M$-test. This completes the proof. — On replacing $x$ by $\pi x$, the development 3) goes over into $(A)$.

Corresponding developments for the tangent can be obtained, either by the same method or by a change of variable, $x = \frac{\pi}{2} - x'$; or still again by means of the identity:

$$\tan x = \cot x - 2\cot 2x.$$

Thus:

4) $$\frac{\pi}{2}\tan\frac{\pi x}{2} = \sum_{n=0}^{\infty} \frac{2x}{(2n+1)^2 - x^2},$$

5) $$\tan x = \frac{2x}{\left(\frac{\pi}{2}\right)^2 - x^2} + \frac{2x}{\left(\frac{3\pi}{2}\right)^2 - x^2} + \frac{2x}{\left(\frac{5\pi}{2}\right)^2 - x^2} + \cdots.$$

The identity:

$$\frac{1}{\sin x} = \cot\frac{x}{2} - \cot x$$

leads to the development:

6) $$\frac{\pi}{\sin \pi x} = \frac{1}{x} - \frac{2x}{1 - x^2} + \frac{2x}{2^2 - x^2} - \frac{2x}{3^2 - x^2} + \cdots.$$

And now a change of variable gives:

7) $$\frac{\pi}{\cos\frac{\pi x}{2}} = 4\left[\frac{1}{1 - x^2} - \frac{3}{3^2 - x^2} + \frac{5}{5^2 - x^2} - \cdots\right].$$

*Linear Denominators.* In one respect these formulas are lacking in simplicity: the denominators are factorable. If we had tried to use Equation 1) instead of Equation 2), we should have arrived at a series whose terms are

$$\frac{1}{x-k\pi}.$$

But the series

$$\sum_k \frac{1}{x-k\pi}, \qquad \sum_k \frac{1}{x+k\pi}$$

do not converge.

It is possible to obviate the difficulty by writing:

$$\frac{2x}{x^2-k^2\pi^2} = \left[\frac{1}{x-k\pi}+\frac{1}{k\pi}\right]+\left[\frac{1}{x+k\pi}-\frac{1}{k\pi}\right].$$

Thus

$$8)\quad \cot x = \frac{1}{x}+\sum_{n=1}^{\infty}\left[\frac{1}{x-n\pi}+\frac{1}{n\pi}\right]+\sum_{n'=1}^{\infty}\left[\frac{1}{x+n'\pi}-\frac{1}{n'\pi}\right].$$

The introduction of the additive term $1/k\pi$ or $-1/k\pi$ seems artificial. How did we come to think of it? A satisfactory answer can be given if we start, not with the cotangent, but with the square of the cosecant. Proceeding as before we find that

$$\frac{1}{\sin^2 x} = \frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{1}{(x-n\pi)^2}+\sum_{n'=1}^{\infty}\frac{1}{(x+n'\pi)^2}$$

and this equation can be written in the simpler form:

$$9)\qquad \csc^2 x = \sum_{n=-\infty}^{\infty}\frac{1}{(x-n\pi)^2}.$$

Consider any interval,

$$-A \leqq x \leqq A.$$

The series:

$$\sum_{n=m}^{\infty}\frac{1}{(x-n\pi)^2}, \qquad \sum_{n'=m}^{\infty}\frac{1}{(x+n'\pi)^2}, \qquad A < m\pi,$$

converge uniformly in this interval, and their terms are continuous Hence

$$\int_0^x \sum_{n=m}^{\infty}\frac{1}{(x-n\pi)^2}\,dx = \sum_{n=m}^{\infty}\int_0^x \frac{dx}{(x-n\pi)^2} =$$

$$-\sum_{n=m}^{\infty}\left[\frac{1}{x-n\pi}+\frac{1}{n\pi}\right],$$

with a similar expression for the other integral. Now transpose the omitted terms in 9), so that the left hand side becomes:

$$\csc^2 x - \sum_{n=-m+1}^{m-1} \frac{1}{(x-n\pi)^2}.$$

That indefinite integral of this function, which approaches 0 as $x$ approches 0, must be equal to the sum of the two integrals we have just considered. Thus we arrive in a perfectly natural manner at the development 8). It is customary to write it in the form:

$$10) \qquad \cot x = \frac{1}{x} + \sum_{n=-\infty}^{\infty}{}' \left[ \frac{1}{x-n\pi} + \frac{1}{n\pi} \right],$$

the prime denoting that the value $n = 0$ is to be omitted.

The second form of the cotangent development is:

$$11) \qquad \pi \cot \pi x = \frac{1}{x} + \sum_{n=-\infty}^{\infty}{}' \left[ \frac{1}{x-n} + \frac{1}{n} \right].$$

These developments have the advantage that each term of the series has a pole at just one point.

## EXERCISES

1. Obtain the development 6) by means of partial fractions.

2. Deduce Taylor's expansion for sine and cosine from de Moivre's Theorem:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

3. Develop the definition and theory of the exponential function on the basis of the limit:

$$E(x) = \lim_{m=\infty} \left(1 + \frac{x}{m}\right)^m.$$

Obtain the functional equation:

$$E(x+y) = E(x)\, E(y)$$

directly from the definition.

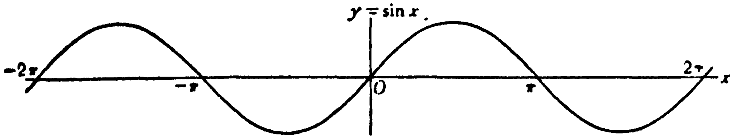
4. Replace each of the other developments by one in which the terms have a pole at only one point.

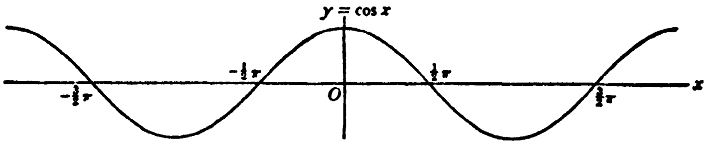
**§6. Infinite Products.** It is now easy to pass to the developments into Infinite Products. Thus

1)
$$\frac{\sin \pi x}{\pi} = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

2)
$$\cos \frac{\pi x}{2} = \prod_{n=1}^{\infty} \left[1 - \frac{x^2}{(2n-1)^2}\right]$$

3)
$$\frac{\sin \pi x}{\pi} = x \prod_{n=-\infty}^{\infty} \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}}$$





# Chapter VII

## Algebraic Transformations of Infinite Series

**§ 1. Elementary Theorems.** We have defined an Infinite Series as the *mark* (Chap. 1, § 1)

1) $$u_0 + u_1 + \cdots.$$

It *converges* when the variable

2) $$s_n = u_0 + u_1 + \cdots + u_{n-1}$$

approaches a limit, $n = \infty$.

We pointed out that the series 1) converges when the series

3) $$u_m + u_{m+1} + \cdots$$

converges, $m$ being any fixed number; and conversely. Also, that, when 1) converges, the series

4) $$k u_0 + k u_1 + \cdots$$

converges, $k$ being any constant.

A *necessary* condition for the convergence of the series 1) is, that $\lim u_n = 0$, $n = \infty$. And a *sufficient* condition for convergence is, that the series of absolute values,

$$| u_0 | + | u_1 | + \cdots,$$

converge.

All this before the $\epsilon$-definition of a limit had been formulated. It is hard to know how systematic to be with these early theorems. They are like the propositions with which the old school books in Elementary Geometry used to begin: "All straight angles are equal." They puzzle the beginner by their obviousness, and are useless to the advanced student as formulated laws, since he automatically thinks in terms of them. We will try to be complete without seeming to be pedantic.

### Exercise

Examine each of the general theorems of convergence which appear in Chap. 1 and give an $\epsilon$-proof, or refer the proof to such theorems as you have already proved by $\epsilon$'s.

**§ 2. The Commutative Law.** If a series

$$u_1 + u_2 + \cdots$$

converges absolutely, its terms may be rearranged at pleasure and the new series will converge to the same value as the old.

For, let

$$s_n = u_1 + \cdots + u_n,$$

and suppose first that $0 \leqq u_n$. Let the rearranged series be

$$u_1' + u_2' + \cdots$$

with

$$s_n' = u_1' + \cdots + u_n'.$$

If the value of the first series be denoted by $U$, then

$$s_n' \leqq U.$$

Hence the second series converges, and its value

$$U' \leqq U.$$

Now interchange the series, regarding the first as a rearrangement of the second. Then

$$U \leqq U',$$

and so $U' = U$.

In the general case, let

$$s_n = \sigma_p - \tau_q,$$

as in Chap. 1, § 9. Let

$$s_n' = \sigma_{p'}' - \tau_{q'}'.$$

Then, by the result just established,

$$\lim \sigma_p = \lim \sigma_{p'}', \qquad \lim \tau_q = \lim \tau_{q'}',$$

and the proof is complete.

If, on the other hand, the given series converges conditionally, its terms can be rearranged so that the new series will converge to an arbitrarily preassigned value, $A$. For, take positive terms till the sum exceeds $A$; then negative terms till the sum falls below $A$; then positive terms again; and so on. The process can be continued indefinitely, since the variables $\sigma_p$, $\tau_q$ diverge monotonically toward $+\infty$. Moreover, since $\lim u_n = 0$, the new series will converge toward $A$.

The terms can evidently also be so rearranged that the new series will diverge toward $+\infty$ or $-\infty$, or oscillate at pleasure. We can state the result as follows.

THEOREM. *Let*

$$u_1 + u_2 + \cdots$$

*be an absolutely convergent series, and let*

$$u_1' + u_2' + \cdots$$

*be a series made up of the $u_n$'s rearranged in any manner. Then the latter series converges absolutely to the same value.*

### EXERCISES

1. Show that, if every series formed out of the terms of a given series converges, the given series converges absolutely.
2. Show that the terms of a conditionally convergent series can be rearranged without altering the value of the series, provided no term is moved more than $k$ places, where $k$ is a constant.

## §3. The Associative Law.

Let

$$u_1 + u_2 + \cdots$$

be a convergent series. Then it is possible to insert parentheses at pleasure:

$$(u_1 + \cdots + v_{m_1}) + (u_{m_1+1} + \cdots + u_{m_2}) + \cdots$$

and the new series will converge to the same value as the old. For, if

$$s_n = u_1 + \cdots + u_n$$

and if $s_k'$ is the sum of the first $k$ terms of the second series, then the $s_k'$ are merely an infinite subset of the $s_n$'s. Hence:

THEOREM 1. *In a convergent infinite series, the terms may be grouped in parentheses in any manner and the new series will converge to the same value as the old.*

The converse is not true. Consider the series of parentheses

$$(1\tfrac{1}{2} - 1\tfrac{1}{3}) + (1\tfrac{1}{4} - 1\tfrac{1}{5}) + \cdots$$

This series converges; but if the parentheses are removed, the general term of the new series does not even approach 0, and so the series diverges. We can, however, state the following theorem.

**Theorem 2.** *Let*

$$\sum_k (u_{m_k} + u_{m_k+1} + \cdots + u_{m_{k+1}})$$

*be a series of parentheses. If, on removing the parentheses, the new series converges, then the old one does, also, and the values are the same.*

The proof follows at once from Theorem 1.

**Theorem 3.** *Consider a convergent series of parentheses. If the number of terms in a parenthesis remains less than a fixed number and if the general term in the parenthesis approaches* 0, *the parentheses may be removed; i.e. the new series will converge to the value of the old one.*

### Exercises

1. If
$$a_1 + a_2 + \cdots$$
and
$$b_1 + b_2 + \cdots$$
are convergent series, show that the series
$$a_1 + b_1 + a_2 + b_2 + \cdots$$
converges, and that its value is equal to the sum of the two given series.

2. Extend the result to the sum of $p$ series.

**§4. Double Series.** By a *double series* is meant the array

$$\begin{array}{c} u_{11} + u_{12} + u_{13} + \cdots \\ u_{21} + u_{22} + u_{23} + \cdots \\ u_{31} + u_{32} + u_{33} + \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array} \tag{1}$$

where $u_{mn}$ is any function of the two natural numbers, $m$ and $n$. Another form of the double series is:

$$\sum_{m,n} u_{mn}.$$

The series is defined to be *convergent* if every simple series formed out of its terms converges. Thus if a double series converges, it necessarily converges absolutely.

This is the definition which is useful in the theory of functions. In special branches of analysis, like the developments for which double Fourier's series are charatceristic, other definitions are expedient.

It is obvious that the series 1) converges if there is a convergent double series

$$\sum_{m,n} v_{mn}, \qquad 0 \leqq v_{mn},$$

such that

$$|u_{mn}| \leqq v_{mn}.$$

It is sufficient that these inequalities hold for

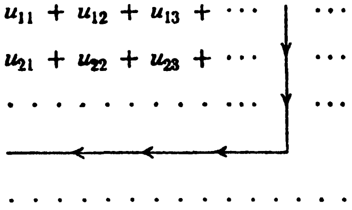
$$M \leqq m + n,$$

where $M$ is some fixed number.

The terms of a convergent double series can be rearranged at pleasure. In particular, the rows and columns may be interchanged.

Of the many ways in which a convergent double series may be evaluated, two are especially important:

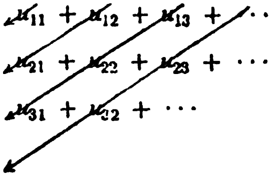
i) Summation by Rectangles:


$u_{11} + u_{12} + u_{13} + \cdots$ | $\cdots$
$u_{21} + u_{22} + u_{23} + \cdots$ | $\cdots$
$\cdots\cdots\cdots$ | $\cdots$
$\cdots\cdots\cdots\cdots\cdots$

where the sides increase together in any manner — for example, always equal:

$$u_{11} + u_{12} + u_{22} + u_{21} + u_{13} + u_{23} + \cdots$$

ii) Summation by Diagonals:


$u_{11} + u_{12} + u_{13} + \cdots$
$u_{21} + u_{22} + u_{23} + \cdots$
$u_{31} + u_{32} + \cdots$

*Example.* Consider the double series:

$$\begin{array}{l} 1 + x + x^2 + \cdots \\ x + x^2 + x^3 + \cdots \\ x^2 + x^3 + \cdots \\ \cdots\cdots\cdots\cdots \end{array}$$

If we sum by diagonals, we are led to the series:

$$1 + 2x + 3x^2 + \cdots$$

This series converges absolutely for all values of $-1 < x < 1$, and diverges for all other values of $x$. Hence the double series converges for $-1 < x < 1$.

The following criterion for convergence is useful in practice.

Test for Convergence. *Let*

$$\begin{array}{l} v_{11} + v_{12} + v_{13} + \cdots \\ v_{21} + v_{22} + v_{23} + \cdots \\ v_{31} + v_{32} + \cdots \\ \cdots\cdots\cdots\cdots \end{array}$$

*be a double series whose terms are all positive or zero:* $0 \leqq v_{np}$. *If*

i) *the rows converge:*

$$v_n = v_{n1} + v_{n2} + \cdots$$

ii) *the series of the values of the rows converges:*

$$V = v_1 + v_2 + \cdots,$$

*then the double series converges.*

*Proof.* Denote by $S_A$ the sum of the terms occupying any area $A$ in the scheme of the double series. Then

$$S_A \leqq V.$$

For, choose $n$ large enough so that $A$ will be contained in the first $n$ rows. Let

$$s_n = v_1 + v_2 + \cdots + v_n.$$

Then

$$S_A \leqq s_n$$

and

$$s_n \leqq V.$$

Hence

$$S_A \leqq V,$$

and the theorem is proved.

THEOREM. *Let*

$$\begin{array}{l} u_{11} + u_{12} + u_{13} + \cdots \\ u_{21} + u_{22} + u_{23} + \cdots \\ u_{31} + u_{32} + \cdots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \end{array}$$

*be a convergent double series. Then*

i) *the rows converge absolutely:*

$$u_n = u_{n1} + u_{n2} + \cdots;$$

ii) *the series of values of the rows converges absolutely:*

$$U = u_1 + u_2 + \cdots$$

iii) *$U$ is equal to the value of the double series.*

*Proof.* The truth of i) is obvious. Next, denote the value of the double series by $U'$. Then, in particular,

$$| U' - s_{np} | < \epsilon, \qquad \nu \leqq n, \ \rho \leqq p,$$

where $s_{np}$ is the sum of the first $p$ terms from the first $n$ rows.

Allow $p$ to become infinite, and let

$$s_n = u_1 + \cdots + u_n.$$

Then

$$| U' - s_n | \leqq \epsilon, \qquad \nu \leqq n.$$

Thus ii) and iii) are established, and the proof is complete, except for the absolute convergence under ii). But this property appears at once on comparison with the double series of absolute values.

The process of evaluating the given double series by means of the series $\Sigma\, u_n$ may be suggestively described as "summing by rows." Since the terms in a double series may be rearranged at pleasure without affecting the convergence, it follows that the series

$$u_{2p} + u_{1p} + \cdots$$

converges — denote its value by $v_p$ — and furthermore the series

$$v_1 + v_2 + \cdots$$

converges to the value $U$ of the double series. This process may be suggestively described by the words: "summing by columns". We may enunciate, then, the following theorem.

COROLLARY. *A convergent double series may be summed by columns as well as by rows.*

*Multiple Series in General.* All of the foregoing definitions and theorems admit immediate generalization to multiple series, $n > 2$:

$$\sum v_{m_1 m_2 \cdots m_n},$$

where $m_1, m_2, \cdots, m_n$ independently run through the natural numbers.

*Product of Two Series.* Cauchy's theorem relating to the product of two series can now be proved with ease. Let

$$u_1 + u_2 + \cdots,$$
$$v_1 + v_2 + \cdots$$

be two absolutely convergent series. Denote their values by $U$, $V$. Then the product is represented by the following absolutely convergent series:

$$UV = u_1 v_1 + u_1 v_2 + u_2 v_1 + u_1 v_3 + \cdots$$

*Proof.* Form the double series:

$$\begin{array}{l} u_1 v_1 + u_1 v_2 + \cdots \\ u_2 v_1 + u_2 v_2 + \cdots \\ \cdots\cdots\cdots\cdots \end{array}$$

This series converges. For, the series of absolute values:

$$\begin{array}{l} |u_1|\,|v_1| + |u_1|\,|v_2| + \cdots \\ |u_2|\,|v_1| + |u_2|\,|v_2| + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

converges by the Test for Convergence.

Returning to the first series we see that the $n$-th row has the value $u_n V$. The series of values of the rows thus becomes:

$$u_1 V + u_2 V + \cdots,$$

the value of which is $UV$. Now, this is also the value of the double series. On summing the double series diagonally, the product theorem results.

*Extension of the Theorem.* It is not true that any two convergent series can be multiplied in this way, for the resulting series may diverge, as is shown by examples. Still, it is true that if the resulting series converges, it converges to the value that is the product of the values of the given series. For, form the power series:

$$f(x) = u_1 x + u_2 x^2 + \cdots$$
$$\varphi(x) = v_1 x + v_2 x^2 + \cdots$$

By Abel's Theorem, Chap. V, § 6, each converges uniformly in the closed interval $0 \leqq x \leqq 1$; and now the same is true of the product series:

$$f(x)\,\varphi(x) = u_1 v_1 x^2 + (u_1 v_2 + u_2 v_1) x^3 + \cdots$$

This proves the theorem.

**§ 5. Series of Series.** Let a convergent series be given whose terms are sums:

1) $$\sum_{k=1}^{m} u_{k1} + \sum_{k=1}^{m} u_{k2} + \cdots$$

Then this series can be written as the sum of $m$ series:

2) $$\sum_{n=1}^{\infty} u_{1n} + \sum_{n=1}^{\infty} u_{2n} + \cdots + \sum_{n=1}^{\infty} u_{mn},$$

provided each of these latter series converges. In other words:

3) $$\sum_{n=1}^{\infty} \sum_{k=1}^{m} u_{kn} = \sum_{k=1}^{m} \sum_{n=1}^{\infty} u_{kn}.$$

Suppose, however, that the terms of the given series are infinite series:

4) $$\sum_{k=1}^{\infty} u_{k1} + \sum_{k=1}^{\infty} u_{k2} + \cdots.$$

The analogue of the sum 2) is now the infinite series:

5) $$\sum_{n=1}^{\infty} u_{1n} + \sum_{n=1}^{\infty} u_{2n} + \cdots,$$

and the question is: Will the series 5) be equal to the series 4)? — provided, of course, that all the series involved converge.

It clarifies the situation to formulate it as a double-limit question: — Will

6) $$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{mn} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}?$$

That the convergence alone is not enough, is suggested by all our experience with double limits and is proved by a simple example. Consider the double series for which

$$s_{mn} = \frac{n}{m+n},$$

where $s_{mn}$ denotes the sum of the terms in the first $m$ rows and the first $n$ columns. The double limit on the left of 6) converges to the value

$$\lim_{m=\infty} s_{mn} = 0.$$

The double limit on the right of 6) converges to the value

$$\lim_{n=\infty} s_{mn} = 1.$$

A sufficient condition for an affirmative answer is given by the following theorem.

THEOREM. *A sufficient condition that the value of the infinite series whose terms are themselves infinite series:*

$$\sum_{m=1}^{\infty} u_{m1} + \sum_{m=1}^{\infty} u_{m2} + \cdots$$

*be given by the series of series:*

$$\sum_{n=1}^{\infty} u_{1n} + \sum_{n=1}^{\infty} u_{2n} + \cdots,$$

*all the series involved being convergent, is that the double series:*

$$\begin{array}{l} u_{11} + u_{12} + u_{13} + \cdots \\ u_{21} + u_{22} + u_{23} + \cdots \\ u_{31} + u_{32} + \cdots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

*converge.*

That the condition is not necessary, seems likely and is proved by the example:

$$\begin{array}{llll} \frac{1}{2} & -\frac{1}{3} & & \\ & \frac{1}{3} & -\frac{1}{4} & \\ & & \frac{1}{4} & -\frac{1}{5} \end{array}$$

the terms not lying in the two diagonals being all 0. The example is due to Mr. E. J. Moulton.

*Pons asinorum.* At this stage of analysis writers and students are prone to a mistake that might seem to be an individual matter were it not for the facts of experience to the contrary. By a confusion of ideas the following theorem is developed:

"If the series which form the terms of 4) converge absolutlely, and if the series 4) converges absolutely, then the series 4) is equal to the series 5)."

This theorem is false, as is shown by the following example due to Arndt:

$$\left(-\frac{1}{2}\cdot\frac{1}{2}\right)+\left(\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{3}\cdot\frac{2}{3}\right)+\left(\frac{1}{3}\cdot\frac{2}{3}-\frac{1}{4}\cdot\frac{3}{4}\right)+\cdots$$

$$\left(-\frac{1}{2}\cdot\frac{1}{2^2}\right)+\left(\frac{1}{2}\cdot\frac{1}{2^2}-\frac{1}{3}\,\frac{2^2}{3^2}\right)+\left(\frac{1}{3}\cdot\frac{2^2}{3^2}-\frac{1}{4}\cdot\frac{3^2}{4^2}\right)+\cdots$$

$$\left(-\frac{1}{2}\cdot\frac{1}{2^3}\right)+\left(\frac{1}{2}\cdot\frac{1}{2^3}-\frac{1}{3}\,\frac{2^3}{3^3}\right)+\left(\frac{1}{3}\cdot\frac{2^3}{3^3}-\frac{1}{4}\cdot\frac{3^3}{4^3}\right)+\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Nevertheless, this false theorem is rediscovered by each new generation of students and writers, and there is no cure known to man.

**§6. Power Series.** THEOREM. *Let $f(y)$ be a function given by the power series:*

1) $$f(y) = b_0 + b_1 y + b_2 y^2 + \cdots, \qquad -S < y < S;$$

*and let $\varphi(x)$ be a function given by the power series:*

2) $$\varphi(x) = a_0 + a_1 x + a_2 x^2 + \cdots, \qquad -R < x < R.$$

*Let $-S < a_0 < S$. Let*

3) $$y = \varphi(x),$$

*where $x$ is restricted to a certain neighborhood of the point $x = 0$. Thus*

4) $$f(y) = f[\varphi(x)]$$

*becomes a function of $x$ in this neighborhood.*

*Then the function $f[\varphi(x)]$ can be expressed as a power series in $x$, the coefficients of which are found as follows. Let each term in 1) be expressed as a power series in $x$:*

5) $$b_n y^n = a_0^{(n)} + a_1^{(n)} x + a_2^{(n)} x^2 + \cdots$$

*and now let these power series be added as if they were polynomials:*

6) $$f[\varphi(x)] = \sum_{n=0}^{\infty} a_0^{(n)} + x\sum_{n=0}^{\infty} a_1^{(n)} + x^2\sum_{n=0}^{\infty} a_2^{(n)} + \cdots$$

The proof by means of double series is immediate. Form the double series whose $n$-th row is the power series 5):

$$\text{7)} \qquad \begin{array}{cccc} a_0 & 0 & 0 & \\ b_1 a_0 & b_1 a_1 x & b_1 a_2 x^2 & \cdots \\ a_0^{(2)} & a_1^{(2)} x & a_2^{(2)} x^2 & \cdots \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \end{array}$$

This double series converges. For, consider the power series

$$\text{8)} \qquad |a_0| + |a_1| X + |a_2| X^2 + \cdots, \qquad X = |x|.$$

Denote its value by $Y$ and form the power series:

$$\text{9)} \qquad |b_n| Y^n = A_0^{(n)} + A_1^{(n)} X + A_2^{(n)} X^2 + \cdots$$

The double series whose $n$-th row is the series 9) converges, if $X$ be suitably resticted, and

$$|a_k^{(n)}| \leqq A_k^{(n)}.$$

Hence 7) converges.

A case of especial importance in practice is that in which $a_0 = 0$. The coefficients of $x$ in the expansion are then series which break off with a finite number of terms.

*Example* 1. Consider the function:

$$\frac{1}{\sqrt{1 - 2\mu x + x^2}}.$$

It can be represented by a power series in $x$, convergent throughout a certain interval. For, let

$$f(y) = \frac{1}{\sqrt{1 - y}},$$

$$y = \varphi(x) = 2\mu x - x^2$$

Then

$$f(y) = (1 - y)^{-\frac{1}{2}} = 1 + \frac{1}{2} y + \frac{1 \cdot 3}{2 \cdot 4} y^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} y^3 + \cdots$$

The powers of $y$ are here polynomials in $x$, and the coefficients of the early terms in the expansion are readily computed Let

$$\frac{1}{\sqrt{1 - 2\mu x + x^2}} = P_0(\mu) + P_1(\mu) x + P_2(\mu) x^2 + \cdots$$

The coefficients $P_n(\mu)$ are known as *Legendre's polynomials* or *Zonal harmonics*. We find at once:

$$P_0(\mu) = 1, \qquad P_1(\mu) = \mu, \qquad P_2(\mu) = \tfrac{1}{2}(3\mu^2 - 1),$$
$$P_3(\mu) = \tfrac{1}{2}(5\mu^3 - 3\mu), \qquad P_4(\mu) = \tfrac{1}{8}(35\mu^4 - 30\mu^2 + 3).$$

The example is unsatisfactory in two respects, and seems herewith to disparage the theorem. First, there is no indication how far the region of convergence of the series extends; and secondly the law of the coefficients, i. e. the polynomial $P_n(\mu)$ for an arbitrary $n$, is not revealed by the theorem. True; but these are not questions the theorem undertook to settle. The theorem has given us the definition of the $P_n(\mu)$ which lies nearest to the application of these functions, and moreover has provided us with an extremely simple means of computing the early ones.

The example just discussed suggests the relation of the theorem to the Cauchy-Taylor Theorem in the theory of functions of a complex variable. The latter theorem tells us not only that the composite function can be developed into a power series, but it tells us also just how far that series converges, and it gives an explicit determination of the coefficients. The advantage of the present theorem is, that it affords a more convenient means of finding the early coefficients, and sometimes the law of the series.

It is not true in general that, if the terms of a series can be developed by Taylor's Theorem, and if the function represented by the series can also be so developed, then the coefficients in the latter development can be obtained as in the theorem. For example, the function

$$f(x) = \frac{\pi}{2}x$$

can be developed into a Fourier's series, Chap. VIII, §§ 1, 7, 8:

$$f(x) = \frac{\sin x}{1} - \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \cdots,$$

and each term can be developed into a power series; but the latter cannot be obtained by forming the series of like powers of $x$.

*Example* 2. The function $\cot x$.

$$\cot x = \frac{\cos x}{\sin x} = \frac{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots}{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}$$

$$= \frac{1}{x}\,\frac{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots}{1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \cdots}.$$

Here, we can develop the function

$$\frac{1}{1 - \frac{1}{6}x^2 + \frac{1}{120}x^3 - \cdots}$$

by setting

$$y = \tfrac{1}{6}x^2 - \tfrac{1}{120}x^4 + \cdots$$

and applying the geometric expansion:

$$\frac{1}{1-y} = 1 + y + y^2 + \cdots$$

Then multiply the two power series together. This shows that there is an expansion of the form:

$$\cot x = \frac{1}{x} + a_0 + a_1 x + a_2 x^2 + \cdots;$$

but there is a simpler way of obtaining the early coefficients.

It is easy to show that the quotient of any two convregent power series:

$$\frac{a_0 + a_1 x + a_2 x^2 + \cdots}{b_0 + b_1 x + b_2 x^2 + \cdots}, \qquad b_0 \neq 0,$$

can be found by dividing the one series by the other just as if they were polynomials; cf. inf. Exs. 9-11. Applying that method here we have:

$$\begin{array}{rl}
 & 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 \cdots \\ \cline{2-2}
1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \cdots & 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \\
 & 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \cdots \\ \cline{2-2}
 & \quad - \frac{1}{3}x^2 + \frac{1}{30}x^4 - \cdots \\
 & \quad - \frac{1}{3}x^2 + \frac{1}{18}x^4 - \cdots \\ \cline{2-2}
 & \qquad\qquad - \frac{1}{45}x^4 - \cdots
\end{array}$$

Hence

$$\cot x = \frac{1}{x} - \tfrac{1}{3}x - \tfrac{1}{45}x^3 + \cdots$$

## Exercises

Compute the coefficients in the following expansions through the term of the fifth or sixth dimension — or higher, if convenient.

1. $\sin^{-1}(k\sin x) = kx + \frac{k(k^2-1)}{6}x^3 + \frac{9k^5 - 10k^3 + k}{120}x^5 + \cdots.$

2. $$\log\cos x.$$

Suggestion: Set

$$y = -\frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

3. $$\sqrt{\cos x}.$$

4. $$\log(1+e^x).$$

5. $$(1+x^5)/(1+x+x^2).$$

6. $$\tan x.$$

7. Show that

$$\log\sin x = \log x - \tfrac{1}{6}x^2 - \tfrac{1}{180}x^4 - \tfrac{1}{2835}x^6 + \cdots.$$

8. Compute:

$$\lim_{x=0}\frac{8\log\cos x + 8 - 8\cos x + x^4}{2\tan x - 2\sin x - x^3}.$$

9. Show that the quotient of two convergent power series,

$$\frac{a_0 + a_1 x + a_2 x^2 + \cdots}{b_0 + b_0 x + b_2 x^2 + \cdots}, \qquad b_0 \neq 0,$$

can be written as a power series:

$$c_0 + c_1 x + c_2 x^2 + \cdots.$$

10. Determine the coefficients $c_n$ in Question 9 by setting

$$a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= (b_0 + b_1 x + b_2 x^2 + \cdots)(c_0 + c_1 x + c_2 x^2 + \cdots),$$

multiplying out, and equating coefficients.

11. Prove that the values of the $c_0, c_1, c_2, \cdots$ as found in Question 10 are the same as those obtained by long division.

12. Let the terms of the series

$$\sum_{n=1}^{\infty}\left[\frac{1}{x+n}-\frac{1}{n}\right]$$

be developed into power series in $x$:

$$\frac{1}{x+n}-\frac{1}{n}=-\frac{x}{n^2}+\frac{x^2}{n^3}-\cdots, \qquad -1<x<1.$$

Show that the value of the given series is expressed by the series

$$-x\sum_{n=1}^{\infty}\frac{1}{n^2}+x^2\sum_{n=1}^{\infty}\frac{1}{n^3}-\cdots$$

13. Show that the function represented by the series:

$$a_1\sin x+a_2\sin 2x+\cdots,$$

where $|a_n|<\gamma^n$ and $\gamma$ is a positive constant $<1$, can be developed into a power series in $x$.

14. Show that the function represented by the series

$$c_1\,y\,\mathrm{sh}\,x+c_2\,y^2\,\mathrm{sh}\,2x+\cdots,$$

where the coefficients $c_n$ are bounded, is analytic at the origin; cf. § 9.

**§ 7. Bernoulli's Numbers.** The coefficients in expansions like those considered in Chap. VI, § 5 admit simple expression in terms of a set of numbers defined as follows. Consider the function:

$$\frac{x}{2}\,\frac{e^x+1}{e^x-1}.$$

It can be expressed as a power series, and since it is an even function, only the terms of even degree will enter. Let us write, then,

$$1)\qquad \frac{x}{2}\,\frac{e^x+1}{e^x-1}=A+B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}+B_3\frac{x^6}{6!}-\cdots$$

The coefficients $B_1$, $B_2$, $B_3$, $\cdots$ are known as *Bernoulli's Numbers.*

It is easy to obtain a recursion formula for computing them. Equation 1) is equivalent to the following:

$$\frac{x}{2}\left(2+\frac{x}{1}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots\right)$$

$$=\left(\frac{x}{1}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots\right)\left(A+B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}+\cdots\right).$$

On multiplying out and equating coefficients we find:

$$A = 1;$$

$$\frac{1}{2}\cdot\frac{1}{n!} = \frac{A}{(n+1)!} + \frac{B_1}{2!(n-1)!} - \frac{B_2}{4!(n-3)!} + \cdots.$$

The sum on the right ends with

$$\frac{(-1)^{\mu+1}B_\mu}{(2\mu)!\,2!} \qquad \text{when} \quad n = 2\mu + 1;$$

$$\frac{(-1)^{\mu+1}B_\mu}{(2\mu)!} \qquad \text{when} \quad n = 2\mu.$$

The first few $B$'s have the following values:

$$B_1 = \tfrac{1}{6}, \qquad B_2 = \tfrac{1}{30}, \qquad B_3 = \tfrac{1}{42},$$

$$B_4 = \tfrac{1}{30}, \qquad B_5 = \tfrac{5}{66}, \qquad B_6 = \tfrac{691}{2730}.$$

*The Sum* $\Sigma n^r$. To find the sum of the $r$-th powers of the first $n$ natural numbers:

$$1^r + 2^r + \cdots + n^r.$$

It is convenient to develop the formula for $n - 1$ instead of $n$. Let

$$y = 1 + e^x + e^{2x} + \cdots + e^{(n-1)x}.$$

Then

$$\left(\frac{d^r y}{dx^r}\right)_{x=0} = 1^r + 2^r + \cdots + (n-1).^r$$

On the other hand,

$$y = \frac{e^{nx} - 1}{e^x - 1} = \frac{e^{nx} - 1}{x}\,\frac{x}{e^x - 1},$$

and

$$\frac{x}{e^x - 1} = \frac{x}{2}\,\frac{e^x + 1}{e^x - 1} - \frac{x}{2}$$

$$= 1 - \frac{x}{2} + B_1\frac{x^2}{2!} - B_2\frac{x^4}{4!} + B_3\frac{x^6}{6!} - \cdots$$

Hence

$$y = \frac{nx + \dfrac{n^2x^2}{2!} + \dfrac{n^3x^3}{3!} + \cdots}{x}\left(1 - \frac{x}{2} + B_1\frac{x^2}{2!} - B_2\frac{x^4}{4!} + B_3\frac{x^6}{6!} - \cdots\right)$$

$$= A_0 + A_1x + A_2x^2 + \cdots + A_rx^r + \cdots$$

We see, then, that

$$\left(\frac{d^r y}{dx^r}\right)_{x=0} = r!\,A_r.$$

On the other hand $A_r$ can be computed by multiplying together the two power series and comparing the coefficients:

$$A_r = \frac{n^{r+1}}{(r+1)!} - \frac{1}{2}\,\frac{n^r}{r!} + \frac{B_1}{2!}\,\frac{n^{r-1}}{(r-1)!} - \cdots$$

The final formula is:

$$1^r + 2^r + \cdots + (n-1)^r =$$

$$\frac{n^{r+1}}{r+1} - \frac{1}{2}n^r + \frac{B_1}{2!}\,r\,n^{r-1} - \frac{B_2}{4!}\,r(r-1)(r-2)\,n^{r-3} + \cdots$$

§ 8. **The Development of cot x.** From the development of Chap. VI, § 5:

$$1)\qquad \pi \cot \pi x = \frac{1}{x} + \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{x+n} - \frac{1}{n}\right]$$

it is possible to derive a development in terms of a power series, § 7, Ex. 12:

$$2)\qquad \pi \cot \pi x = \frac{1}{x} - 2x\sum_{n=1}^{\infty}\frac{1}{n^2} - 2x^3\sum_{n=1}^{\infty}\frac{1}{n^4} - \cdots$$

On the other hand the cotangent can be expressed in the complex domin as

$$\pi \cot \pi x = \pi\,\frac{\cos \pi x}{\sin \pi x} = \pi i\,\frac{e^{i\pi x} + e^{-i\pi x}}{e^{i\pi x} - e^{-i\pi x}}$$

$$= \frac{1}{x}\;\frac{2\,i\,\pi x}{2}\;\frac{e^{2i\pi x} + 1}{e^{2i\pi x} - 1}$$

This last expression is the basis of the definition of Bernoulli's Numbers, § 7. Thus we find:

$$3)\qquad \pi \cot \pi x = \frac{1}{x}\left[1 - B_1\frac{(2\pi x)^2}{2!} - B_2\frac{(2\pi x)^4}{4!} - \cdots\right].$$

Similar expansions hold for $\tan x$, $\sec x$, $\csc x$.

*The Series* $\sum \frac{1}{n^{2p}}$. A comparison of the series 2) and 3) leads to an evaluation of the series of negative even powers of the integers. We have:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{2!} B_1$$

. . . . . . .

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p}} = \frac{2^{2p-1}\pi^{2p}}{(2p)!} B_p$$

## §9. Analytic Functions of Several Variables.

A function of $n$ variables, $f(x_1, \cdots, x_n)$, is said to be *analytic in the point* $(a_1, \cdots, a_n)$ if it can be developed into a power series:

$$1) \quad f(x_1, \cdots, x_n) = \Sigma\, c_{m_1, \ldots, m_n} (x_1 - a_1)^{m_1} \cdots (x_n - a_n)^{m_n}$$

which converges for a point $(\xi_1, \cdots \xi_n)$ such that

$$\xi_k - a_k \neq 0, \qquad k = 1, \cdots, n.$$

For simplicity, let $n = 2$ and write:

$$2) \quad f(x, y) = c_{00} + c_{10}\, x + c_{01}\, y + c_{20}\, x^2 + c_{11}\, x y + c_{02}\, y^2 + \cdots.$$

If this series converges for a point $(\xi, \eta)$, and if $\xi \neq 0$, $\eta \neq 0$, then it evidently converges for all points $(x, y)$ such that

$$3) \qquad |x| \leqq |\xi|, \qquad |y| \leqq |\eta|.$$

If the coefficients are bounded, $|c_{mn}| \leqq G$, then the series converges at least when

$$4) \qquad |x| < 1, \qquad |y| < 1.$$

The domain of convergence of a power series in several variables is not simple. Thus the series

$$1 + x y + x^2 y^2 + x^3 y^3 + \cdots$$

converges in every point $(x, y)$ such that

$$|x y| < 1,$$

and diverges for all other points. Hence the points of convergence are those which lie on the convex side of both hyperbolas:

$$x y = 1, \qquad x y = -1,$$

A point $(\xi, \eta)$ is said to be *interior* to the region of convergence if the series converges for a point

$$|\xi| + \alpha, \qquad |\eta| + \beta$$

where $\alpha$, $\beta$ are both positive.

If the series 2) converges in the point $(x, y)$, then each of the series

$$\sum_{n=1}^{\infty} c_{mn}\, y^n$$

converges, and

$$5) \qquad f(x,y) = \sum_{n=1}^{\infty} c_{0n}\, y^n + x \sum_{n=1}^{\infty} c_{1n}\, y^n + \cdots.$$

But the converse is not true; the convergence of 5) does not insure the convergence of 2).

A convergent power series can be differentiated term-by-term at any point interior to the region of convergence, and the derivative series will admit at least the same region of convergence as the original series. It follows, then, that the coefficients in a convergent power series are given by Maclaurin's Theorem:

$$6) \quad f(x,y) = f(0,0) + f_1(0,0)\,x + f_2(0,0)\,y + \tfrac{1}{2} f_{20}(0,0)\,x^2 + \cdots.$$

The theorem of § 7 admits immediate extension to power series in several variables. As a consequence we have the following important property of power series.

If $(\xi, \eta)$ is an interior point of the region of convergence of the power series 2), then $f(x,y)$ can be represented by a convergent power series:

$$f(\xi + h, \eta + k) = \sum c'_{mn}\, h^m k^n$$

throughout a certain neighborhood of the point $(\xi, \eta)$. If follows, then, that if a function is analytic at a point, it is analytic at every interior point of the region of convergence of the series. The domain of definition of the function may now be extended by the process of *analytic continuation* familiar in the theory of functions of one and of several complex variables.

Taylor's Theorem for an arbitrary interior point of the region of convergence of the power series now follows at once:

$$\begin{aligned} f(x,y) &= f(x_0, y_0) + f_{10}(x_0, y_0)(x - x_0) + f_{01}(x_0, y_0)(y - y_0) \\ &+ \frac{1}{2!} f_{20}(x_0, y_0)(x - x_0)^2 + \frac{2}{2!} f_{11}(x_0, y_0)(x - x_0)(y - y_0) \\ &+ \frac{1}{2!} f_{02}(x_0, y_0)(y - y_0)^2 + \cdots. \end{aligned}$$

*Analytic Curve, Surface, etc.* A curve:

$$x_k = x_k(t), \qquad k = 1, \cdots, n,$$

is said to be *analytic at a point* $(x_1^0, \cdots, x_n^0)$ if each of the functions $x_k(t)$ is analytic at the point $t = t_0$ and $x_k^0 = x_k(t_0)$; and if, furthermore, not all the derivatives $x_k'(t)$ vanish there. An arc of a curve is said to be *analytic* if it is analytic at each of its points.

A surface:

$$x_k = x_k(u, v), \qquad k = 1, \cdots, n,$$

is said to be *analytic at a point* $(x_1^0, \cdots, x_n^0)$ if each of the functions $x_k(u, v)$ is analytic at the point $(u_0, v_0)$ and $x_k^0 = x_k(u_0, v_0)$; and if, moreover, the rank of the matrix

$$\left\| \begin{matrix} \dfrac{\partial x_1}{\partial u} \cdots \dfrac{\partial x_n}{\partial u} \\ \dfrac{\partial x_1}{\partial v} \cdots \dfrac{\partial x_n}{\partial v} \end{matrix} \right\|$$

at $(u_0, v_0)$ is 2. A piece of surface is said to be *analytic* if it is analytic at each of its points.

The extension of the definition to manifolds of higher order is obvious.

### EXERCISES

1. Let

$$u = \frac{1}{r}$$

$$r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2,$$

$(a, b, c) \neq (0, 0, 0)$, $r > 0$. Show that the function $u$ is analytic in the origin.

2. Let

$$u = f(x, y, z)$$

be analytic in the origin. Then

$$f(x, y, z) = u_0(x, y, z) + u_1(x, y, z) + \cdots,$$

where $u_n(x, y, z)$ is a homogeneous polynomial of degree $n$, or is 0. Furthermore, let $u$ be harmonic:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Show that $u_n(x, y, z)$ is harmonic.

**§ 10. Regular Curves. Jordan Curves.** We have defined *analytic curves*, and now mathematical perspective demands some mention of the more general concept of *curve*, even though a detailed analytical developement of these ideas lies outside the frame of these Lectures.

Let

1) $$x = f(t), \qquad y = \varphi(t),$$

where $f(t)$, $\varphi(t)$ are continuous in the closed interval

2) $$0 \leqq t \leqq 1.$$

Under suitable further restrictions Equations 1) will represent a curve in the $(x, y)$-plane. Let us first, however, consider the possibilities when no further restrictions are imposed.

a) The point set $\{(x, y)\}$ defined by 1) may consist of a *single point*. This is the case when $f(t)$, $\varphi(t)$ are both constants.

b) At the other extreme, the point set $\{(x, y)\}$ defined by 1) may fill a two-dimensional region of the $(x, y)$-plane. Peano has given an example in which every point of a *square* in the $(x, y)$-plane belongs to the point set $\{(x, y)\}$. It is only fair to say, however, that the transformation of the interval 1) on the square is not one-to-one. Some points of the square are obtained more than once.

In each of these examples the transformation of the points of the interval 2) on the elements of the point set $\{(x, y)\}$ defined by 1) has failed to be one-to-one. Let us, then, with Jordan, seize on this requirement as the further restriction to be imposed on the functions $f(t)$, $\varphi(t)$ and say :— If $x'$, $y'$ are any two numbers such that the equations

3) $$x' = f(t), \qquad y' = \varphi(t)$$

admit a common root $t = t'$, where $0 < t' < 1$, then these equations 3) shall admit no further root $t = t''$, where $0 \leqq t'' \leqq 1$.

Such a point set $\{(x, y)\}$ is known as a *Jordan curve*. If, in particular,

4) $$f(0) = f(1), \qquad \varphi(0) = \varphi(1),$$

the curve is *closed*, and it is a fundamental theorem of *analysis situs* that such a curve divides the plane into an interior region and an exterior region.

But if Equations 4) are not both true, the end points of the curve are distinct, and we have an *open* curve. The curve is *simple*, i.e. has no multiple points. For otherwise the equations 3) would admit two distinct solutions, $t'$ and $t''$, in the closed interval 2).

Returning to the case of a closed curve we observe that there is a one-to-one and continuous relation between the points of such a curve and the points of a circle.

A Jordan curve may fail to have a tangant at each and every point. Moreover, there is another important property which it fails to share with the curves we ordinarily think of, like arcs of ellipses, cycloids, etc. or even arbitrary analytic arcs (or regular arcs, cf. infra). Any one of these latter arcs can be embedded in a two-dimensional region of arbitrarily small area. But it may be impossible to enclose a Jordan curve in a two-dimensional open set, the area of which is less than an arbitrary positive quantity — even when the "area" is measured in the sense of Lebesgue.

*Regular Curves.* Consider an open Jordan curve. Impose on $f(t)$, $\varphi(t)$ the still further restrictions that each of these functions possess a continuous first derivative in the closed interval 2). Such a Jordan curve is defined as an *arc of a regular curve*, or as a *regular arc.*

A *regular curve* is now defined as the point set made up by stringing together a finite number of regular arcs, $C_1, C_2, \cdots, C_n$; the terminal point of $C_k$ coinciding with the initial point of $C_{k+1}$, $k = 1, 2, \cdots, n-1$.

A regular curve may be open or closed. It is not necessarily simple. It may have a whole arc of multiple points. Or it may have an infinite number of multiple points, all but one of which are isolated; e.g.

$$\begin{cases} y = x^3 \sin \dfrac{1}{x}, & 0 < x \leqq 1; \\ y = 0, \ x = 0, & \end{cases}$$

is a regular arc, and

$$y = 0, \qquad 0 \leqq x \leqq 1,$$

is another. Together they make up a regular curve.

### EXERCISE

Show that a point set $\{(x, y)\}$ consisting in part of a two-dimensional open set can never be mapped in a one-to-one manner and continuously on a line segment, 2).

# Chapter VIII

## Fourier's Series

**§1. Fourier's Series.** By a *trigonometric* or *Fourier's series* is meant a series of the form:

$$1) \qquad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

If the series converges and, after being multiplied by $\cos kx$ or $\sin kx$, can be integrated term by term, it is easy to determine the coefficients. Observe the formulas of integration:

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0;$$

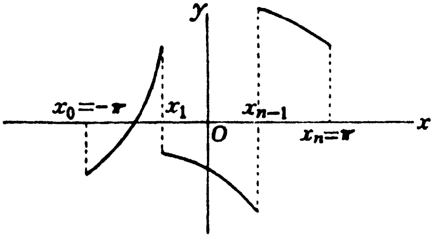
$$\left.\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \\ \int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \end{aligned}\right\} 0, \; m \neq n; \qquad \pi, \; m = n.$$

Thus we find, on denoting the value of the series by $f(x)$:

$$2) \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

These numbers, $a_n$ and $b_n$, are known as the *Fourier's Coefficients* of the function $f(x)$. They have a meaning for any continuous function, and for a great variety of discontinuous functions, aside altogether from the problem of whether the function can be developed into a Fourier's series or not. Let us study some of their properties.

We shall restrict ourselves to functions $f(x)$ which are continuous in the interval $(-\pi, \pi)$ except at most at a finite number of points, $x_1, \cdots, x_{n-1}$; cf. figure on next page. Within each interval $(x_k, x_{k+1})$ the function shall have a continuous derivative, and both function and derivative shall approach limits at each extremity of the sub-interval. In a point $x_k$,



$$f(x_k) = \tfrac{1}{2}\{ f(x_k^+) + f(x_k^-) \},$$

where

$$f(c^+) = \lim_{x=c+} f(x), \qquad f(c^-) = \lim_{x=c-} f(x).$$

Moreover,

$$f(\pi) = \tfrac{1}{2}\{ f(-\pi^+) + f(\pi^-) \}.$$

Thus $f(x)$ is defined in the interval $-\pi < x \leqq \pi$. For all other values of $x$ it shall be defined by the requirement of periodicity:

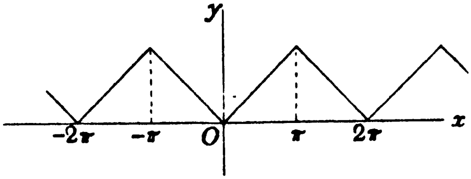
$$f(x + 2\pi) = f(x).$$

We shall prove in §§ 7, 8 that such a function can be developed into a Fourier's series.

## EXERCISES

1. Let $f(x)$ be defined as follows:

$$f(x) = |x|, \qquad -\pi < x \leqq \pi:$$
$$f(x + 2\pi) = f(x).$$



Show that, if the function can be developed,

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right).$$

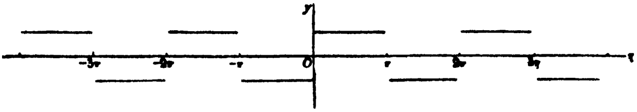
2. Let $f(x)$ be defined as follows:

$$f(x) = 1, \qquad 0 < x < \pi;$$

$$f(x) = -1, \qquad -\pi < x < 0;$$

$$f(0) = 0, \qquad f(\pi) = 0;$$

$$f(x + 2\pi) = f(x).$$



Show that, if $f(x)$ can be developed,

$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

3. If $f(x)$ is an even function:

$$f(-x) = f(x),$$

show that only the cosine terms will appear:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad a_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\,dx;$$

and if $f(x)$ is an odd function:

$$f(-x) = -f(x),$$

only the sine terms will appear:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\,dx$$

4. If $f(x)$ is defined merely in the interval

$$0 \leqq x \leqq \pi,$$

show that $f(x)$ can be developed into a sine series, and also that $f(x)$ can be developed into a cosine series.

5. If $f(x)$ is defined merely in the interval

$$0 \leqq x \leqq \pi,$$

and if

$$f(\pi - x) = f(x),$$

show that, if $f(x)$ be developed into a cosine series or a sine series, only alternate terms will appear:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n} \cos 2nx,$$

$$f(x) = \sum_{n=1}^{\infty} b_{2n-1} \sin (2n-1)x.$$

6. Assuming that the function

$$f(x) = \cos \alpha x$$

can be developed into a Fourier's series, convergent in the interval $-\pi \leqq x \leqq \pi$, show that

$$\cos \alpha x = \frac{\sin \alpha \pi}{\pi} \left\{ \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha \cos n\pi \cos nx}{\alpha^2 - n^2} \right\}.$$

On setting $x = \pi$ the development becomes:

$$\cos \alpha \pi = \frac{\sin \alpha \pi}{\pi} \left\{ \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2} \right\}.$$

Changing the notation, we have the development:

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}.$$

§2. **Bessel's Inequality.** *Normal Functions.* Let $\varphi_0(x)$, $\varphi_1(x), \cdots, \varphi_n(x)$ be a set of functions, each continuous in a closed interval $a \leqq x \leqq b$, and such that

$$3) \qquad \int_a^b \varphi_m(x)\,\varphi_n(x)\,dx = 0, \qquad m \neq n.$$

Moreover, they shall be linearly independent. Such a system is called a set of *orthogonal functions*. As an example:

$$4) \qquad \varphi_0(x) = \tfrac{1}{2}, \qquad \varphi_{2n}(x) = \cos nx, \qquad \varphi_{2n-1}(x) = \sin nx.$$

It is in terms of orthogonal functions that the most important developments of mathematical physics take place:

$$5) \qquad f(x) = c_0\,\varphi_0(x) + c_1\,\varphi_1(x) + \cdots.$$

If such a development is possible, and if on multiplying through by $\varphi_n(x)$ it is possible to integrate term by term, the coefficients will be given by the formula:

$$6)\qquad c_n = \frac{\int_a^b f(x)\,\varphi_n(x)\,dx}{\int_a^b [\varphi_n(x)]^2\,dx}.$$

Observe that the denominator cannot vanish, since no function $\varphi_n(x)$ can vanish identically.

If $\varphi_0(x)$, $\varphi_1(x)$, $\cdots$ is any set of orthogonal functions, then $\alpha_0\,\varphi_0(x)$, $\alpha_1\,\varphi_1(x)$, $\cdots$, where the $\alpha_n$'s are any positive constants, is also an orthogonal system. It is obvious that the $\alpha_n$ can be so chosen that

$$7)\qquad \int_a^b [\varphi_n(x)]^2\,dx = 1.$$

The modified set of functions is said to be *normalized*. Thus the set:

$$8)\qquad \frac{1}{\sqrt{2\pi}},\qquad \frac{1}{\sqrt{\pi}}\cos nx,\qquad \frac{1}{\sqrt{\pi}}\sin nx,\qquad -\pi < x < \pi,$$

is an example of a normal system. Equation 6) then becomes:

$$9)\qquad c_n = \int_a^b f(x)\,\varphi_n(x)\,dx.$$

*Approximation by a Normal Polynomial.* How can we use the $n$ coefficients $c_k$ most advantageously, in order best to approximate to a given function, $f(x)$, by a polynomial of normal functions,

$$10)\qquad c_0\,\varphi_0(x) + c_1\,\varphi_1(x) + \cdots + c_{n-1}\,\varphi_{n-1}(x)\quad ?$$

The question can, of course, be answered in many ways. One of the most useful answers is that given by Bessel, who used the idea of Least Squares and demanded that the $c_k$'s be so determined as to make the integral of the square of the error:

$$u = \int_a^b [f(x) - c_0\,\varphi_0(x) - \cdots - c_{n-1}\,\varphi_{n-1}(x)]^2\,dx$$

a minimum. A necessary condition for a minimum is, that

$$\frac{\partial u}{\partial c_k} = -2\int_a^b [f - c_0\varphi_0 - \cdots - c_{n-1}\varphi_{n-1}]\varphi_k\,dx = 0.$$

Since the $\varphi_k(x)$ form a normal set, this equation reduces to the following:

$$c_k = \int_a^b f(x)\,\varphi_k(x)\,dx. \tag{11}$$

These are the same values as those given by 9), and furthermore it is to be noted that a given $c_k$ does not change as $n$ increases.

If these values be substitued in the integral, the latter reduces to

$$\int_a^b [f(x)]^2\,dx - c_0^2 - c_1^2 - \cdots - c_{n-1}^2\,.$$

But the integral is never negative, and so we arrive at *Bessel's Inequality*:

$$c_0^2 + c_1^2 + \cdots + c_{n-1}^2 \leqq \int_a^b [f(x)]^2\,dx. \tag{12}$$

From 12) we infer that the series

$$c_0^2 + c_1^2 + \cdots \tag{13}$$

converges. In particular, we see that the series formed from the squares of the Fourier's coefficients converge:

$$\left\{\begin{array}{l} a_0^2 + a_1^2 + a_2^2 + \cdots \\ \qquad b_1^2 + b_2^2 + \cdots \end{array}\right. \tag{14}$$

## EXERCISES

1. Let $\varphi(x)$ be continuous in the closed interval $a \leqq x \leqq b$. Show that each of the integrals

$$\int_a^b \varphi(x)\cos nx\,dx, \qquad \int_a^b \varphi(x)\sin nx\,dx$$

approaches the limit 0 as $n$ becomes infinite.

Suggestion. Begin with the case that $a = -\pi$, $b = \pi$.

2. Extend the theorem of Question 1 to the case that $\varphi(x)$ is continuous in the above closed interval except for a finite number of points, and $\varphi(x)$ is absolutely integrable; i. e. the integral

$$\int_a^b |\varphi(x)|\,dx$$

converges.

Suggestion. Isolate the points of discontinuity by arbitrarily small neighborhoods. Show that the contribution of these neighborhoods is uniformly small for all values of $n$. Then apply the theorem of Question 1 to each of the remaining intervals.

3. If $\varphi(x)$ satisfies the conditions of Question 2, show that

$$\lim_{n=\infty}\int_a^b \varphi(x)\sin(n+\tfrac{1}{2})\,x\,dx = 0.$$

## § 3. Appraisal of the Fourier's Coefficients.

If a function $f(x)$ of the class here considered (§ 1) has no discontinuities, the series

$$15)\qquad \left\{\begin{aligned} a_0 + a_1 + a_2 + &\cdots \\ b_1 + b_2 + &\cdots \end{aligned}\right.$$

converge absolutely, and so the Fourier's series converges absolutely and uniformly for all values of $x$.

The proof is as follows. Transform the integrals in 2) by integration by parts. Since $f(x)$ is continuous and periodic, we find:

$$16)\qquad \left\{\begin{aligned} \int_{-\pi}^{\pi} f(x)\cos nx\,dx &= -\frac{1}{n}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx \\ \int_{-\pi}^{\pi} f(x)\sin nx\,dx &= \frac{1}{n}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx. \end{aligned}\right.$$

Let the Fourier's coefficients of the function $f'(x)$ be denoted by primes:

$$17)\qquad a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx, \qquad b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx.$$

Thus the relations 16) can be written in the form:

$$18)\qquad a_n = -\frac{b_n'}{n}, \qquad b_n = \frac{a_n'}{n}.$$

We cannot integrate again by parts without making requirements respecting the second derivative of $f(x)$. We can, however, attain our ends by means of an algebraic device. It is obvious that

$$0 \leqq \left(|b_n'| - \frac{1}{n}\right)^2.$$

Hence

$$2\frac{|b_n'|}{n} \leqq b_n'^2 + \frac{1}{n^2}.$$

But the series

$$b_1'^2 + b_2'^2 + \cdots$$

converges by § 2, 14); and the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots$$

also converges. From 18), then,

$$2\,|a_n| \leqq b_n'^2 + \frac{1}{n^2},$$

and hence the first of the series 15) converges absolutely. — The convergence of the second series is shown in a similar manner.

**§ 4. Identical Vanishing.** We come now to a theorem which is proved with equal ease for a more general class of functions than that of § 1.

THEOREM. *Let $\varphi(x)$ be continuous in the interval*

$$-\pi < x \leqq \pi,$$

*except at most for a finite number of points; and let the function be bounded. Let all the Fourier's coefficients vanish:*

$$\int_{-\pi}^{\pi} \varphi(x) \cos nx\, dx = 0, \qquad \int_{-\pi}^{\pi} \varphi(x) \sin nx\, dx = 0.$$

*Then $\varphi(x) = 0$, except in the points of discontinuity.*

*Proof.* If the theorem is not true, let $x = \lambda$ be an interior point of the interval, in which $\varphi(x)$ is continuous and $\neq 0$; — positive, say: $\varphi(\lambda) > 0$. Then there exists an interval $(a, b)$:

$$-\pi < a \leqq x \leqq b < \pi,$$

where

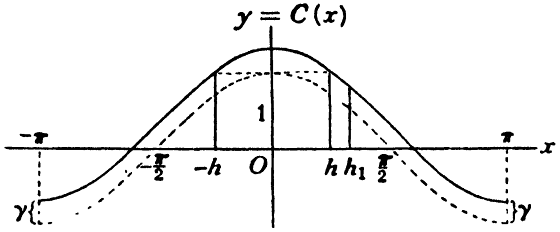
$$\lambda = \frac{a+b}{2},$$

such that $\varphi(x)$ is positive throughout $(a, b)$, and is, therefore, greater than a certain positive constant, $M$:

$$M < \varphi(x), \qquad a \leqq x \leqq b.$$

Let $h_1 < \pi/2$ be so chosen that the interval $|x - \lambda| \leqq h_1$ lies in $(a, b)$, and choose $0 < h < h_1$. Form the function

$$C(x) = \gamma + \cos x$$



and determine $\gamma$ so that

$$C(h) = 1,$$

or

$$\gamma = 1 - \cos h.$$

Then $C(x)$ has these two properties:

i) $$1 \leqq C(x), \qquad |x| \leqq h;$$

ii) $$-r \leqq C(x) \leqq r,$$

$$-\pi \leqq x \leqq -h_1, \qquad h_1 \leqq x \leqq \pi,$$

where $r$ is the larger of the two constants $1 - \gamma$ and $C(h_1)$, and hence $0 < r < 1$.

And now we apply this function as follows. Let

$$y = C(x - \lambda).$$

Consider the integral:

$$\int_{-\pi}^{\pi} \varphi(x)\, y^n \, dx,$$

where $n = 2m$ is an even natural number. If $m$ is chosen large enough, the value of this integral will be positive. For, the integral can be written as:

$$\int_{\lambda-h_1}^{\lambda+h_1} \varphi(x)\, y^n\, dx$$

$$+ \int_{-\pi}^{\lambda-h_1} + \int_{\lambda+h_1}^{\pi}.$$

The first integral is greater than

$$\int_{\lambda-h}^{\lambda+h} \varphi(x)\, y^n dx > 2\, M h$$

for all values of $m$.

Let $G$ be the upper limit of $|\varphi(x)|$ in the interval $(-\pi, \pi)$. Furthermore, if $\epsilon > 0$ is arbitrarily chosen, $m$ can then be so determined that $r^n < \epsilon$, and consequently

$$y^n < \epsilon, \qquad -\pi \leqq x \leqq \lambda - h_1 \quad \text{or} \quad \lambda + h_1 \leqq x \leqq \pi.$$

Hence the second and third integrals, taken together, are less than

$$2\, \pi\, G\, \epsilon.$$

Thus

$$\int_{-\pi}^{\pi} \varphi(x)\, y^n\, d\, x > 2\, M\, h - 2\, \pi\, G\, \epsilon.$$

Let $\epsilon$ be so chosen that the right hand side is positive.

Here is a contradiction, for

$$\int_{-\pi}^{\pi} \varphi(x)\, y^n\, d\, x = 0$$

for all values of $n$. In fact,

$$y^n = [\, \gamma + \cos(x - \lambda)\,]^n$$

can be written as a polynomial in $\sin x$, $\cos x$, and then converted into a trigonometric polynomial, cf. § 5:

$$y^n = \sum_{k=0}^{n} (A_k \cos k\, x + B_k \sin k\, x).$$

Hence

$$\int_{-\pi}^{\pi} \varphi(x)\, y^n\, dx = \sum_{k=0}^{n} \left\{ A_k \int_{-\pi}^{\pi} \varphi(x) \cos kx\, dx \right.$$

$$\left. + B_k \int_{-\pi}^{\pi} \varphi(x) \sin kx\, dx \right\}.$$

But the Fourier's coefficients of $\varphi(x)$ all vanish by hypothesis. — This completes the proof.

**§ 5. The Formulas of Summation.** The following formulas of summation are especially useful in the study of Fourier's series.

$$\text{A)} \begin{cases} \frac{1}{2} + \cos\varphi + \cos 2\varphi + \cdots + \cos n\varphi = \dfrac{\sin \frac{2n+1}{2}\varphi}{2 \sin \frac{\varphi}{2}} \\[2ex] \sin\varphi + \sin 2\varphi + \cdots + \sin n\varphi = \dfrac{\cos\frac{\varphi}{2} - \cos\frac{2n+1}{2}\varphi}{2 \sin \frac{\varphi}{2}} \end{cases}$$

$$\text{B)} \begin{cases} \frac{1}{2} + \cos\varphi + \cos 2\varphi + \cdots + \cos n\varphi = \dfrac{\cos n\varphi - \cos(n+1)\varphi}{2(1 - \cos\varphi)} \\[2ex] \sin\varphi + \sin 2\varphi + \cdots + \sin n\varphi = \dfrac{\sin\varphi + \sin n\varphi - \sin(n+1)\varphi}{2(1-\cos\varphi)} \end{cases}$$

These formulas can be deduced most expeditiously by the aid of complex quantities from the geometric progression:

$$1 + e^{\varphi i} + e^{2\varphi i} + \cdots + e^{n\varphi i} = \frac{e^{(n+1\varphi)i} - 1}{e^{\varphi i} - 1}.$$

They can, however, once given, be established by the method of mathematical induction.

Similar formulas for

$$\cos\varphi - \cos 2\varphi + \cos 3\varphi - \cdots$$

$$\sin\varphi - \sin 2\varphi + \sin 3\varphi - \cdots$$

can be obtained by replacing $\varphi$ by $\varphi + \pi$. The denominator will be $2 \cos\frac{\varphi}{2}$ or $2(1 + \cos\varphi)$.

*Other Formulas of Trigonometry.* We mention de Moivre's Theorem:

$$\text{C)}\quad \begin{cases} \cos m\varphi = \cos^m\varphi - \dfrac{m(m-1)}{1\cdot 2}\cos^{m-2}\varphi\sin^2\varphi \\ \qquad\qquad + \dfrac{m(m-1)(m-2)(m-3)}{1\cdot 2\cdot 3\cdot 4}\cos^{m-4}\varphi\sin^4\varphi + \cdots \\ \sin m\varphi = m\cos^{m-1}\varphi\sin\varphi - \dfrac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cos^{m-3}\varphi\sin^3\varphi \\ \qquad\qquad + \cdots \end{cases}$$

proved most easily by expanding the binomial on the right of the equation

$$e^{m\varphi i} = (\cos\varphi + i\sin\varphi)^m,$$

and equating coefficients; also by mathematical induction.

Furthermore, a quasi inverse:

$$\text{D)}\qquad \cos^p x\sin^q x = \sum_{k=0}^{m}(A_k\cos kx + B_k\sin kx), \qquad p + q = m.$$

This formula is likewise readily established by induction.

### EXERCISES

1. Prove that

$$\cos\varphi + \cos 3\varphi + \cdots + \cos(2n-1)\varphi = \frac{\sin 2n\varphi}{2\sin\varphi},$$

$$\sin\varphi + \sin 3\varphi + \cdots + \sin(2n-1)\varphi = \frac{1-\cos 2n\varphi}{2\sin\varphi}.$$

2. Obtain formulas for

$$\cos\varphi - \cos 3\varphi + \cdots + (-1)^{n+1}\cos(2n-1)\varphi,$$

$$\sin\varphi - \sin 3\varphi + \cdots + (-1)^{n+1}\sin(2n-1)\varphi.$$

Suggestion: Write the odd numbers as $4\mu + 1$ and $4\mu - 1$.

3. Prove that

$$(\alpha\cos x + \beta\sin x + \gamma)^m$$

can be written in the form:

$$\sum_{k=1}^{m}(A_k\cos kx + B_k\sin kx).$$

**§6. Abel's Theorem.** If the coefficients of a Fourier's series form a series that converges absolutely, then the Fourier's series converges uniformly for all values of $x$, as is seen from Weierstrass's $M$-Test. In the case of certain other important Fourier's series, the uniform convergence can be established by Abel's theorem, Chap. V, §6. Let $\alpha_1, \alpha_2, \cdots$ be any set of numbers such that

$$\alpha_1 \geqq \alpha_2 \geqq \alpha_3 \geqq \cdots, \qquad \lim_{n=\infty} \alpha_n = 0.$$

Then the series

$$\alpha_1 \sin\varphi + \alpha_2 \sin 2\varphi + \cdots$$

converges uniformly in any interval $(a, b)$ which with its extremities lies inside the interval $0 < x < 2\pi$:

$$0 < a \leqq x \leqq b < 2\pi.$$

We wish to prove that, to a positive $\epsilon$ chosen at pleasure, there corresponds an $m$ independent of $\varphi$ such that

$$|\alpha_{m+1}\sin(m+1)\varphi + \cdots + \alpha_{m+p}\sin(m+p)\varphi| < \epsilon,$$

$$p = 1, 2, \cdots$$

Now, by §5, $A$):

$$\sin(m+1)\varphi + \sin(m+2)\varphi + \cdots + \sin(m+p)\varphi$$

$$= \frac{\cos\dfrac{2m+3}{2}\varphi - \cos\dfrac{2m+2p+1}{2}\varphi}{2\sin\dfrac{\varphi}{2}}$$

and so

$$-\frac{1}{\sin\dfrac{\varphi}{2}} \leqq \sin(m+1)\varphi + \cdots + \sin(m+p)\varphi \leqq \frac{1}{\sin\dfrac{\varphi}{2}}$$

for all values of $p$. If, then, we choose $c$ as any positive quantity $< \pi$, we shall have

$$-\frac{1}{\sin\dfrac{c}{2}} \leqq \sin(m+1)\varphi + \cdots + \sin(m+p)\varphi \leqq \frac{1}{\sin\dfrac{c}{2}},$$

$$c \leqq \varphi \leqq 2\pi - c.$$

Hence by Abel's theorem

$$-\frac{\alpha_{m+1}}{\sin\dfrac{c}{2}} \leqq \alpha_{m+1}\sin(m+1)\varphi + \cdots + \alpha_{m+p}\sin(m+p) \leqq \frac{\alpha_{m+1}}{\sin\dfrac{c}{2}},$$

$$c \leqq \varphi \leqq 2\pi - c,$$

and so it is sufficient to choose $m$ so that

$$\frac{\alpha_{m+1}}{\sin\frac{c}{2}} < \epsilon.$$

The same theorem holds for the series

$$\alpha_1 \cos\varphi + \alpha_2 \cos 2\varphi + \cdots.$$

A similar theorem holds for the series

$$\alpha_1 \cos\varphi - \alpha_2 \cos 2\varphi + \alpha_3 \cos 3\varphi - \cdots$$

and

$$\alpha_1 \sin\varphi - \alpha_2 \sin 2\varphi + \alpha_3 \sin 3\varphi - \cdots$$

the interval $(a, b)$ of uniform convergence now lying within the interval $(-\pi, \pi)$:

$$-\pi < a \leqq x \leqq b < \pi.$$

*Examples.*

a) $$\frac{\sin\varphi}{1} + \frac{\sin 2\varphi}{2} + \frac{\sin 3\varphi}{3} + \cdots$$

b) $$\frac{\cos\varphi}{1} + \frac{\cos 2\varphi}{2} + \frac{\cos 3\varphi}{3} + \cdots$$

## EXERCISES

1. Prove that the series

$$\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots$$

converges uniformly in any interval

$$-\pi < a \leqq x \leqq b < 0 \qquad \text{or} \qquad 0 < a \leqq x \leqq b < \pi.$$

2. State and prove a similar theorem for the series:

$$\frac{\cos x}{1} + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \cdots,$$

**§ 7. Proof of Convergence.** We have stated in § 1 the theorem that the Fourier's series corresponding to a function $f(x)$ of the class there defined, converges for all values of the argument and represents the function; i.e. the value of the series is equal, for each $x$, to the value of the function. We will now prove that theorem. Begin with the case of no discontinuities of $f(x)$. Then the Fourier's series converges uniformly for all values of $x$ (§ 3) and so represents a continuous function,

$$1) \qquad F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The Fourier's coefficients of this function can actually be obtained by the method suggested at the beginning of § 1, and so we have:

$$2) \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \sin nx\, dx;$$

i.e. the Fourier's coefficients of the functions $f(x)$ and $F(x)$ are identical.

Form the function:

$$\varphi(x) = f(x) - F(x).$$

This is also a function of the class defined in § 1, and it is continuous without exception. Its Fourier's coefficients are all 0. Hence by § 4 it is 0 for all values of $x$, and the functions $f(x)$, $F(x)$ are seen to be identical:

$$3) \qquad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Thus the theorem is proved for continuous functions.

**§ 8. Continuation. The Discontinuous Case.** *A Special Function.* The proof of § 7 justifies the development of § 1, Exercise 1. But we can go a step further and establish the development of Exercise 2. For, the first series can be differentiated term by term, by the theorem of Chap. V, § 9, at all points at which $f(x)$ has a derivative. In fact, let $x_0$ be any point of the interval $0 < x < \pi$ or $-\pi < x < 0$. It is then possible to include $x_0$ within an interval $(a, b)$ which, with its end points, lies within the interval in question:

$$-\pi < a < x_0 < b < 0 \qquad \text{or} \qquad 0 < a < x_0 < b < \pi.$$

Within this interval the series

$$\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots$$

converges uniformly; cf. § 6, Exercise 1. The other conditions for differentiation term-by-term are satisfied, and thus the developement of Ex. 2 is obtained.

*Oscillation of $f(x)$ at a Point of Discontinuity.* Let

$$D = f(c^+) - f(c^-).$$

Then $D$ is the *oscillation* of $f(x)$ at the point $c$.

Let

$$\Phi(x) = \frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

Then

$$\Phi(x) = \tfrac{1}{2}, \qquad 0 < x < \pi;$$

$$\Phi(x) = -\tfrac{1}{2}, \qquad -\pi < x < 0.$$

Thus the oscillation of $\Phi(x)$ at $x = 0$ is 1.

Form the function:

$$f(x) - D\,\Phi(x - c),$$

where $f(x)$ is a function of the class defined in § 1, and $D$ is its oscillation at $c$. This function also belongs to that class of functions, and its oscillation at $c$ is 0; i.e. it is continuous there.

Let the discontinuities of $f(x)$ in the interval $-\pi < x \leqq \pi$ lie in the points

$$-\pi < c_1 < c_2 < \cdots < c_n \leqq \pi$$

and let the oscillation in $c_k$ be $D_k$. Then the function

$$f(x) - \sum_{k=1}^{n} D_k\,\Phi(x - c_k)$$

will be continuous for all values of $x$, and also belong to the class of functions defined in § 1. It can, therefore, be expanded into a Fourier's series, absolutely and uniformly convergent to the function for all values of $x$.

The function $\Phi(x-c)$ can also be expanded into a Fourier's series whose general term is

$$\frac{\sin(2n-1)(x-c)}{2n-1}=$$

$$\frac{-\sin(2n-1)c}{2n-1}\cos(2n-1)x+\frac{\cos(2n-1)c}{2n-1}\sin(2n-1)x,$$

and we know all about the uniform convergence of this series.

We see, then, that the original function $f(x)$ can be expanded, and the proof is complete.

We remark that the Fourier's expansion of the function,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

can be broken up into the series

$$\sum_{n=1}^{\infty}a_n\cos nx,\qquad \sum_{n=1}^{\infty}b_n\sin nx.$$

For,

$$f(-x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx-b_n\sin nx),$$

and the two series can be added and subtracted.

The same result might have been obtained by observing that any function $f(x)$ of § 1 can be written as the sum of an even and an odd function:

$$f(x)=\tfrac{1}{2}[f(x)+f(-x)]+\tfrac{1}{2}[f(x)-f(-x)].$$

Each of these functions can be developed into a Fourier's series. The first development will contain only cosines, the second only sines.

One further remark. A function $f(x)$ of the class here considered may vanish identically throughout any subinterval of the interval $(-\pi,\pi)$; but its Fourier's coefficients will all vanish if and only if the entire interval is made up of a finite number of such subintervals and their end points; i.e. $f(x)\equiv 0$.

**§ 9. The Gibbs Effect.** When the Fourier's development represents a discontinuous function, the series cannot converge uniformly, since the terms are continuous. One might expect the approximation curves to proceed fairly directly, as in the case of

$$\lim_{n=\infty} x^{\frac{1}{2n-1}}.$$

On the other hand, they might rise to indefinite heights, as in the case of

$$\lim_{n=\infty} n x e^{-nx^2}.$$

They do neither. The function $s_n(x)$ is bounded, but the graph rises appreciably above, and drops appreciably below, the graph of the limiting function near the point of discontinuity. This is the phenomenon which Gibbs first pointed out, and which Bôcher was the first to treat mathematically*. We turn now to a detailed study of the phenomenon, following Bôcher's methods, but using a slightly different series**.

We have seen that the function

$$1)\qquad f(x) = \frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots$$

has the value:

$$f(x) = \begin{cases} \frac{1}{4}\pi, & 2n\pi < x < (2n+1)\pi \\ -\frac{1}{4}\pi, & (2n-1)\pi < x < 2n\pi \\ 0, & x = n\pi \end{cases}$$

We wish to study the approximation curves and, in particular, to show that they remain finite. Let

$$2)\qquad s_n(x) = \frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots + \frac{\sin (2n-1)x}{2n-1}.$$

Then

$$3)\qquad s_n'(x) = \cos x + \cos 3x + \cdots + \cos (2n-1)x = \frac{\sin 2nx}{2 \sin x}.$$

Because of symmetry it is sufficient to study the function in the interval

$$0 \leqq x \leqq \frac{\pi}{2}.$$

We have:

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* Gibbs, *Nature*, vol. 59, 1899, p. 606. Bôcher, *Annals of Mathematics*, 2d. ser., vol. 7 (1906) p. 123.

** This is the series used by Carslaw, *Fourier's Series and Integrals*. On p. 273 there is a carefully drawn graph of the approximation curve for a particular value of *n*.

$$4)\qquad y = s_n(x) = \int_0^x \frac{\sin 2nx}{2\sin x}\,dx.$$

This integral can be transformed as follows. Write:

$$\frac{1}{\sin x} = \frac{1}{x} + 2\varphi(x),$$

where

$$5)\qquad 2\varphi(x) = \frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}, \qquad 0 < x \leqq \frac{\pi}{2}.$$

Moreover, we define

$$\varphi(0) = \lim_{x=0+} \varphi(x) = 0.$$

Thus $\varphi(x)$, together with its derivatives, is continuous in the closed interval $0 \leqq x \leqq \frac{\pi}{2}$.

We can now write:

$$6)\qquad s_n(x) = \int_0^x \frac{\sin 2nx}{2x}\,dx + \int_0^x \varphi(x)\sin 2nx\,dx.$$

The second of these integrals can be appraised by integration by parts. Thus

$$\int_0^x \varphi(x)\sin 2nx\,dx = -\frac{\varphi(x)\cos 2nx}{2n} + \frac{1}{2n}\int_0^x \varphi'(x)\cos 2nx\,dx,$$

Denote the value of this integral by $\Psi$:

$$\Psi(x,n) = \int_0^x \varphi(x)\sin 2nx\,dx.$$

Let $M$ be the maximum value of $|\varphi(x)|$, $|\varphi'(x)|$ in the interval. Then

$$7)\qquad |\Psi(x,n)| \leqq \frac{G}{n}, \qquad G = (\tfrac{1}{2} + \tfrac{1}{4}\pi)M, \qquad 0 \leqq x \leqq \tfrac{\pi}{2}.$$

Hence

$$8)\qquad s_n(x) = \int_0^x \frac{\sin 2nx}{2x}\,dx + \Psi, \quad |\Psi| \leqq \frac{G}{n}, \qquad 0 < x \leqq \tfrac{\pi}{2}.$$

This latter integral can be thrown into more transparent form by a change of variable. Set

9) $$2nx = t, \qquad x = \frac{t}{2n}.$$

Then

10) $$\int_0^x \frac{\sin 2nx}{2x}\,dx = \frac{1}{2}\int_0^t \frac{\sin t}{t}\,dt.$$

This integral converges when $t = \infty$. It is, in fact, well known that

11) $$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2};$$

but we do not need the quantitative result to establish the truth of our statement, that the function $s_n(x)$ is bounded. That proof is now complete, since

12) $$s_n(x) = \frac{1}{2}\int_0^t \frac{\sin t}{t}\,dt + \Psi,$$

and each of these functions is bounded.

*Remark*. The reader who is interested in following the quantitative relations more closely, can readily do so. The ordinates of the curve

13) $$y = s_n(x)$$

can be appraised with an absolute error of less than any preassigned positive $\epsilon$ for large values of $n$ as follows. First, determine $m$ so that

$$\frac{G}{m} \leqq \epsilon.$$

Then the ordinates will be given to the degree of accuracy in question by the integral

$$\int_0^x \frac{\sin 2nx}{2x}\,dx = \frac{1}{2}\int_0^t \frac{\sin t}{t}\,dt.$$

In particular, the maxima and minima of the curve 13) are given by the roots of the function $s_n'(x)$:

$$x_1 = \frac{\pi}{2n}, \qquad x_2 = \frac{\pi}{n},$$

$$x_3 = \frac{3\pi}{2n}, \qquad x_4 = \frac{2\pi}{n},$$

$$\cdots\cdots\cdots\cdots\cdots$$

In any case, $x_k = \frac{k\pi}{2n}$. Thus

$$\int_0^{x_k} \frac{\sin 2nx}{2x}\,dx = \int_0^{k\pi} \frac{\sin t}{t}\,dt$$

$$= \int_0^{\pi} + \int_{\pi}^{2\pi} + \int_{2\pi}^{3\pi} + \cdots + \int_{(k-1)\pi}^{k\pi}$$

The terms in this sum are alternately positive and negative, and steadily decreasing in numerical value, so that the sum can be written in the form:

$$v_0 - v_1 + v_2 - v_3 + \cdots$$

The values of the early $v_n$'s are found to be*:

$$\begin{array}{ll} v_0 = 1.852 & v_4 = 0.142 \\ v_1 = 0.434 & v_5 = 0.116 \\ v_2 = 0.257 & v_6 = 0.098 \\ v_3 = 0.183 & \end{array}$$

**§ 10. Integration and Differentiation of the Expansion.** The series arising from the Fourier's expansion of any function of the class defined in § 1 can be integrated term by term throughout any finite interval. This is obvious when there are no discontinuities, for then the series converges uniformly for all values of $x$, and the terms are continuous. But even when discontinuities are present, integration term by term is possible in any interval which does not contain or abut on a singularity; for in such an interval the convergence is uniform, and the terms are always continuous. Moreover the function

$$s_n(x) = \frac{a_0}{2} + \sum_{n=1}^{n} (a_n \cos nx + b_n \sin nx)$$

is bounded. If, then, the points of discontinuity which lie in an interval of length $2\pi$ be excluded from the interval by arbitrarily small neighborhoods, the contribution of these neighborhoods both to the integral of the function and to the integral of the remainder $r_n(x)$ of the series:

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* cf. Bôcher, l. c., p. 129, where further references are given.—The details of the study here outlined are given by Carslaw, l. c.

$$f(x) = s_n(x) + r_n(x),$$

will be small, and hence

$$\int_a^b f(x)\,dx = \lim_{n=\infty} \int_a^b s_n(x)\,dx,$$

or the integral of the series is equal to the series of the integrals. — It is left to the student to put this proof into $\epsilon$-form.

*Differentiation.* If $f(x)$ is a continuous function of the class defined in § 1, and if its derivative also belongs to this class, then the Fourier's series which represents $f(x)$ can be differentiated term by term. For, the derivative $f'(x)$ can be developed into a Fourier's series, and this series can be integrated term by term. The latter series is a Fourier's development for $f(x)$. But the Fourier's development of $f(x)$ is unique.

A repeated application of this result enables us to establish the following more general theorem.

THEOREM. *If $f(x)$ and its derivatives of the first $n$ orders are all functions of the class defined in* § 1, *and if $f(x)$ and its first $n-1$ derivatives are continuous for all values of $x$, then the Fourier's expansion of $f(x)$ can be differentiated $n$ times term by term.*

Böcher has obtained a number of more general theorems relating to differentiation term by term, as well as to the development into a convergent Fourier's series, in the article cited in § 9. This article is of elementary character and affords an important supplement to the treatment here given. It is based on Poisson's Integral, the analytic treatment of which was subsequently simplified, without however, thanks to a remark of Professor Perkins, losing the advantage of a simple geometric interpretation of the convergence on the boundary*.

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* cf. the Author's *Funktionentheorie*, vol. I, 5th ed. 1928, p. 669.

**§ 11. Divergent Series.** In the ancient days of modern science mathematicians operated with divergent series with a naïveté which lost much of its charm when the more serious requirements of convergence became central in the thought of the age. Let us turn back for a moment and see what were some of the things they did.

Consider the series

$$1) \qquad 1 - 1 + 1 - 1 + \cdots$$

It is no use to sum the first $n$ terms (the men of that time never did!). But this series *must* have a meaning — so simple a series as that cannot be a no-body. Let us call its value $x$:

$$x = 1 - 1 + 1 - 1 + \cdots$$

Well, now we see that

$$x = 1 - (1 - 1 + 1 - 1 + \cdots) = 1 - x$$

Ah ha!

$$x = 1 - x,$$

$$x = \tfrac{1}{2}.$$

I told you it had a meaning, and there it is:

$$\tfrac{1}{2} = 1 - 1 + 1 - 1 + \cdots$$

But you are not convinced by my logic? How banal! However, to humor you, I'll give another proof. You admit with all your modern sophistication that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad 0 < x < 1.$$

And you believe in limits. Very well. Let $x$ approach 1 as its limit. You see now, do you not, that the left-hand side of this equation approaches the limit $\tfrac{1}{2}$ — even Weierstrass would have admitted so much. And every school teacher knows that "if two variables are always equal and each approaches a limit, the limits are equal". So the right-hand side also approaches the limit $\tfrac{1}{2}$, and again we have:

$$\tfrac{1}{2} = 1 - 1 + 1 - 1 + \cdots$$

The mathematicians of that age may not have been rigorous, but there was a joy of living which the world can never afford to ignore. Their antics still amuse us; they can instruct us. Turn again to the series 1) and be modern: compute the sum of the first $n$ terms. It is alternately 1 and 0:

$$s_1 = 1, \qquad s_2 = 0, \qquad s_3 = 1, \qquad s_4 = 0, \cdots$$

Well, its *average value* is $\frac{1}{2}$, isn't it? That's something to go on. In fact, it is a great deal, for it gave Frobenius as far back as 1880 (Crelle 89) an idea which can be formulated as follows.

*Summable Series.* Let

$$2) \qquad a_1 + a_2 + \cdots$$

be a series, and let

$$3) \qquad s_n = a_1 + a_2 + \cdots + a_n.$$

Take the average of the first $n$ $s_n$'s:

$$4) \qquad S_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

If $S_n$ approaches a limit as $n$ becomes infinite:

$$\lim_{n=\infty} S_n = c,$$

then the series 2) is said to be *summable*, and the number $c$ is attached to it as its *value* (sometimes called its *sum*).

If the series 2) is convergent, then $c$ will be its value in the ordinary sense, or $\lim s_n = c$.

Apply this definition to the series 1):

$$\begin{aligned} S_n &= \tfrac{1}{2} && \text{when} \quad n = 2m \\ &= \frac{m}{2m-1} && \text{when} \quad n = 2m - 1 \end{aligned}$$

Hence

$$\lim_{n=\infty} S_n = \tfrac{1}{2},$$

and so $\frac{1}{2}$ is the value assigned by this definition to the series 1).

But there was a second argument of Friend Euler, why the series 1) must have the value $\frac{1}{2}$. Let us see what that view can do for us. Form the power series

5) $$a_0 + a_1 x + a_2 x^2 + \cdots$$

And now I say: *If the series* 2) *is summable, the series* 5) *will converge when* $|x| < 1$:

6) $$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots.$$

*The function* $f(x)$ *will approach a limit as* $x$ *approaches* 1, *and the value of this limit will be the value,* $c$, *of the summable series* 2):

$$\lim_{x=1^-} f(x) = c.$$

From the definition of $S_n$ it is easy to infer that

$$s_1 = S_1,$$
$$s_2 = 2S_2 - S_1,$$
$$s_3 = 3S_3 - 2S_2,$$

and, generally:

7) $$s_n = nS_n - (n-1)S_{n-1}.$$

From this formula we read off at once the following

LEMMA. *If*

$$|S_n| < G, \quad n = 1, 2, \cdots,$$

*then*

8) $$|s_n| < 2nG.$$

Let

$$s_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1},$$

$$S_n(x) = \frac{s_1(x) + \cdots + s_n(x)}{n}.$$

We proceed to transform $s_n(x)$ by a process similar to that used in the proof of Abel's Theorem, Chap. V, § 6. From 3) we have:

$$a_0 = s_1, \qquad a_1 = s_2 - s_1, \qquad \cdots, \qquad a_{n-1} = s_n - s_{n-1}.$$

Hence

$$s_n(x) = s_1 + (s_2 - s_1)x + \cdots + (s_n - s_{n-1})x^{n-1}$$

$$= \sum_{\nu=1}^{n-1} s_\nu (x^{\nu-1} - x^\nu) + s_n x^{n-1}$$

$$= (1-x)\sum_{\nu=1}^{n-1} s_\nu x^{\nu-1} + s_n x^{n-1}.$$

The infinite series

$$\sum_{\nu=1}^{\infty} s_\nu x^{\nu-1}, \qquad |x| < 1,$$

converges because of 7); and likewise the term $s_\nu x^{\nu-1}$ approaches a limit, namely, 0. Hence the series 5) converges when $|x| < 1$, and

9) $$f(x) = (1 - x)\sum_{\nu=1}^{\infty} s_\nu x^{\nu-1}.$$

*New Proof of Abel's Theorem.* From this last equation we readily infer Abel's Theorem. For, if $\lim s_\nu = c$, then

$$c - \epsilon < s_\nu < c + \epsilon. \qquad m \leqq \nu.$$

Hence

$$(c - \epsilon)\frac{x^{m-1}}{1 - x} \leqq \sum_{\nu=m}^{\infty} s_\nu x^{\nu-1} \leqq (c + \epsilon)\frac{x^{m-1}}{1 - x}$$

and so

$$(c - \epsilon)x^{m-1} \leqq (1 - x)\sum_{\nu=m}^{\infty} s_\nu x^{\nu-1} \leqq (c + \epsilon)x^{m-1}$$

or

$$(1 - x)\sum_{\nu=m}^{\infty} s_\nu x^{\nu-1} = (c + \zeta)x^{m-1}, \qquad |\zeta| \leqq \epsilon.$$

Thus from 9)

$$f(x) = (1 - x)\sum_{\nu=1}^{m-1} s_\nu x^{\nu-1} + (c + \zeta)x^{m-1}.$$

Hence

$$f(x) - c = (1 - x)\sum_{\nu=1}^{m-1} s_\nu x^{\nu-1} - c(1 - x^{m-1}) + \zeta x^{m-1}.$$

In this equation rests the proof. For

$$|\zeta x^{m-1}| < \epsilon,$$

where $m$ is already fixed, and $|x| < 1$. And now each of the first two terms on the right can be made numerically less than $\epsilon$ in absolute value by restricting $x$ to a suitable interval,

$$1 - \delta < x < 1.$$

Hence

$$|f(x) - c| < 3\epsilon, \qquad 0 < 1 - x < \delta.$$

Thus

$$\lim_{x=1-}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n, \qquad \text{q. e. d.}$$

We return now to the proof of the main theorem. From 7) it follows that

$$\sum_{\nu=1}^{m} s_\nu x^{\nu-1} = \sum_{\nu=1}^{m} [\nu S_\nu - (\nu - 1) S_{\nu-1}] x^{\nu-1}$$

$$= \sum_{\nu=1}^{m-1} \nu S_\nu (x^{\nu-1} - x^\nu) + m S_m x^{m-1}$$

$$= (1 - x) \sum_{\nu=1}^{m-1} \nu S_\nu x^{\nu-1} + m S_m x^{m-1}.$$

Since 2) is summable,

$$|S_n| < G,$$

and so the last term on the right approaches 0 when $m = \infty$ ($|x| < 1$). For the same reason, the series

$$\sum_{\nu=1}^{\infty} \nu S_\nu x^{\nu-1}, \qquad |x| < 1,$$

converges. Hence finally

$$f(x) = (1 - x)^2 \sum_{\nu=1}^{\infty} \nu S_\nu x^{\nu-1}. \tag{10}$$

The last step:

$$\lim_{x=1^-} f(x) = c,$$

can now be taken exactly as in the proof of Abel's Theorem just given. Since $S_n$ approaches $c$,

$$c - \epsilon < S_\nu < c + \epsilon, \qquad m \leqq \nu.$$

Hence

$$(c - \epsilon) \sum_{\nu=m}^{\infty} \nu x^{\nu-1} \leqq \sum_{\nu=m}^{\infty} \nu S_\nu x^{\nu-1} \leqq (c + \epsilon) \sum_{\nu=m}^{\infty} \nu x^{\nu-1},$$

or

$$\sum_{\nu=m}^{\infty} \nu S_\nu x^{\nu-1} = (c + \zeta) \sum_{\nu=m}^{\infty} \nu x^{\nu-1}, \qquad |\zeta| \leqq \epsilon. \tag{11}$$

Observe that

$$\frac{1}{(1 - x)^2} = \sum_{\nu=1}^{\infty} \nu x^{\nu-1},$$

and write as a matter of notation:

$$A = \sum_{\nu=1}^{m-1} \nu S_\nu x^{\nu-1}, \qquad B = \sum_{\nu=1}^{m-1} \nu x^{\nu-1}.$$

Thus

$$\sum_{\nu=m} \nu S_\nu x^{\nu+1} = \sum_{\nu=1} \nu S_\nu x^{\nu-1} - A = \frac{f(x)}{(1-x)^2} - A,$$

$$\sum_{\nu=m} \nu x^{\nu-1} \quad = \sum_{\nu=1} \nu x^{\nu-1} \quad - B = \frac{1}{(1-x)^2} - B.$$

Equation 11), multiplied by $(1-x)^2$, now gives, because of 10):

$$f(x) - (1-x)^2 A = (c + \zeta)[1 - (1-x)^2 B].$$

Hence

$$f(x) - c = \zeta + (1-x)^2 A - (c+\zeta)(1-x)^2 B.$$

Since $A$ and $B$ are polynomials, they are bounded in the interval $0 \leqq x < 1$:

$$|A| < M, \qquad |B| < M, \qquad 0 \leqq x < 1.$$

Consequently

$$|(1-x)^2 A - (c+\zeta)(1-x)^2 B| \leqq (1 + |c| + \epsilon) M (1-x)^2.$$

This last quantity is small for values of $x$ near unity, or

$$(1 + |c| + \epsilon) M (1-x)^2 < \epsilon, \qquad 1 - \delta < x < 1.$$

Hence

$$|f(x) - c| < 2\epsilon, \qquad 1 - \delta < x < 1,$$

and this completes the proof.

**§12. Summable Fourier's Series.** Let $f(x)$ be continuous for all values of $x$ and let $f(x)$ have the period $2\pi$. Form the Fourier's coefficients and write down the Fourier's series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

This series does not in general converge. It is, however, summable, to the value $f(x)$, as we will now prove. Let

$$s_n(x) = \frac{a_0}{2} + \sum_{n=1}^{n-1} (a_n \cos nx + b_n \sin nx).$$

Recalling the expression for the Fourier's coefficients, § 1, 2), we see that

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\tfrac{1}{2} + \cos(t-x) + \cos 2(t-x) + \cdots \right.$$
$$\left. + \cos(n-1)(t-x)\right] dt.$$

The series in the bracket can be summed by Formula *B*), § 5, and thus $s_n(x)$ takes the form:

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\cos(n-1)(t-x) - \cos n(t-x)}{1-\cos(t-x)}\,dt.$$

It is now easy to compute the value of

$$S_n(x) = \frac{s_1(x) + \cdots + s_n(x)}{n},$$

We find:

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{1-\cos n(t-x)}{n[1-\cos(t-x)]}\,dt.$$

Make the change of variable:

$$\alpha = \frac{t-x}{2}, \qquad t = x + 2\alpha.$$

Then

1) $$S_n(x) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x+2\alpha)\,\frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha.$$

Lemma. *Let $\varphi(\alpha)$ be continuous in the interval*

$$0 < \alpha \leqq \tfrac{\pi}{2},$$

*and let $\varphi(\alpha)$ approach a limit when $\alpha$ approaches* 0:

$$\lim_{\alpha=0^+} \varphi(\alpha) = \varphi(0^+).$$

*Then*

$$\lim_{n=\infty}\int_0^{\frac{\pi}{2}} \varphi(\alpha)\,\frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha = \frac{\pi}{2}\,\varphi(0^+),$$

*Similarly,*

$$\lim_{n=\infty}\int_{-\frac{\pi}{2}}^{0}\varphi(\alpha)\frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha = \frac{\pi}{2}\varphi(0^-),$$

*if* $\varphi(\alpha)$ *is continuous in the interval*

$$-\tfrac{\pi}{2} \leqq \alpha < 0,$$

*and* $\varphi(\alpha)$ *approaches a limit when* $\alpha$ *approaches* $0$*:*

$$\lim_{\alpha=0^-}\varphi(\alpha) = \varphi(0^-).$$

*Proof.* Observe that

$$\int_0^{\frac{\pi}{2}}\frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha = \frac{\pi}{2}.$$

For, let

$$\sigma_k(x) = \tfrac{1}{2} + \cos x + \cos 2x + \cdots + \cos(k-1)x.$$

Then

$$\frac{\sin^2 n\alpha}{n\sin^2\alpha} = \frac{2}{n}[\sigma_1(2\alpha) + \cdots + \sigma_n(2\alpha)]$$

and

$$\int_0^{\frac{\pi}{2}}\sigma_k(2\alpha)\,d\alpha = \frac{\pi}{4}.$$

Let

$$\varphi(\alpha) = \varphi(0^+) + \zeta.$$

To an arbitrary positive $\epsilon$ corresponds a positive $\delta$ such that

$$|\zeta| < \epsilon, \qquad 0 < \alpha < \delta.$$

On the other hand, $\zeta$ is bounded, and so

$$|\zeta| \leqq M, \qquad 0 < \alpha \leqq \tfrac{\pi}{2}.$$

We can now write:

$$\int_0^{\frac{\pi}{2}}\varphi(\alpha)\frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha = \varphi(0^+)\int_0^{\frac{\pi}{2}}\frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha$$

$$+\int_0^{\delta} \zeta \frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha + \int_{\delta}^{\frac{\pi}{2}} \zeta \frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha.$$

The first integral on the right has the value $\pi/2$. The other two can be appraised as follows.

$$\left|\int_0^{\delta} \zeta \frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha\right| \leqq \int_0^{\delta} |\zeta| \frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha < \epsilon \int_0^{\frac{\pi}{2}} \frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha = \frac{\pi}{2}\epsilon;$$

$$\left|\int_{\delta}^{\frac{\pi}{2}} \zeta \frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha\right| \leqq \int_{\delta}^{\frac{\pi}{2}} |\zeta| \frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha < \frac{\pi M}{2n\sin^2\delta}.$$

Now, $\epsilon$ and $\delta$ are fixed. Choose $m$ so that

$$\frac{\pi M}{2m\sin^2\delta} < \epsilon.$$

Then

$$\left|\int_0^{\frac{\pi}{2}} \varphi(\alpha) \frac{\sin^2 n\alpha}{n\sin^2\alpha}\,d\alpha - \frac{\pi}{2}\varphi(0^+)\right| < \left(\frac{\pi}{2}+1\right)\epsilon, \qquad m \leqq n,$$

and the first part of the lemma is proved. — The second part follows from the first by a mere change of variable, $\beta = -\alpha$.

Turning now to Equation 1) and applying the Lemma, we see that, for an arbitrary choice of $x$, the variable $S_n(x)$ approaches a limit, and

$$\lim_{n=\infty} S_n(x) = f(x).$$

This proves the theorem.

*Generalizations.* It is evident that the above proof applies to a much more general class of functions. In fact, let $f(x)$ be any function which is continuous at a point $x = x'$, and is integrable and is such that the Fourier's coefficients have a meaning, and that moreover the appraisals used in the proof of the Lemma apply. Then the Fourier's series is summable at the point $x'$ to the value $f(x')$.

In particular, if $f(x)$ has only isolated discontinuities, at each of which both the limits

$$\lim_{x=c+} f(x) = f(c^+), \qquad \lim_{x=c-} f(x) = f(c^-)$$

exist, and if

$$f(c) = \tfrac{1}{2}[f(c^+) + f(c^-)],$$

then

$$\lim_{n=\infty} S_n(c) = f(c).$$

*Uniform Summability.* Returning now to the case that $f(x)$ is everywhere continuous, we see at once that $f(x)$ is uniformly continuous for all values of $x$. To a positive $\epsilon$ chosen at pleasure there corresponds, then, a positive $\delta$ independent of $x$, $x'$ such that

$$|f(x) - f(x')| < \epsilon, \qquad |x - x'| < \delta$$

On using this $\delta$ the foregoing proof shows that the Fourier's series is uniformly summable to the value $f(x)$.

**§ 13. Concluding Remarks.** *A Classical Convergence Proof.* Dunham Jackson has given to a classical convergence proof a particularly simple form. Let $f(x)$ be continuous in the interval $(-\pi, \pi)$ except at a finite number of points, and let $f(x)$ be absolutely integrable there; i. e.

$$\int_{-\pi}^{\pi} |f(x)|\, dx$$

converges. Let $f(x)$ have the period $2\pi$:

$$f(x + 2\pi) = f(x).$$

At a point $x = \xi$ let

i)
$$\lim_{x=\xi+} f(x) = f(\xi^+),$$

and let the difference-quotient

ii)
$$\frac{f(\xi + h) - f(\xi^+)}{h}, \qquad 0 < h < \delta,$$

be bounded.

Consider the Fourier's series of this function. Let

1) $$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

Then

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) [ \tfrac{1}{2} + \cos(t - x) + \cos 2(t - x) + \cdots$$

$$+ \cos n(t - x) ]\, dt.$$

Let

$$\sigma_n(x) = \tfrac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx.$$

Then

$$\sigma_n(x) = \frac{\sin \frac{2n+1}{2} x}{2 \sin \frac{x}{2}}.$$

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, \sigma_n(t - x)\, dt.$$

Set $u = t - x, \qquad t = u + x.$

Then

2) $$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u + x)\, \sigma_n(u)\, du.$$

Break this integral up into

3) $$\int_{-\pi}^{0} + \int_{0}^{\pi}$$

and consider

4) $$\frac{1}{\pi} \int_{0}^{\pi} f(u + x)\, \sigma_n(u)\, du.$$

Observe that

$$\frac{1}{\pi} \int_{0}^{\pi} \sigma_n(u)\, du = \frac{1}{2}.$$

Hence

5) $$\frac{1}{\pi} \int_{0}^{\pi} f(\xi^+)\, \sigma_n(u)\, du = \frac{1}{2} f(\xi^+).$$

Thus

$$6)\qquad \frac{1}{\pi}\int_0^\pi f(u+\xi)\,\sigma_n(u)\,du - \frac{1}{2}f(\xi^+)$$

$$= \frac{1}{\pi}\int_0^\pi \frac{f(u+\xi)-f(\xi^+)}{u}\,\frac{u}{2\sin\frac{1}{2}u}\,\sin(n+\tfrac{1}{2})\,u\,du.$$

This last integral approaches 0 as $n$ becomes infinite; § 2, Exercise 3. Hence the left-hand side of 6) approaches 0, and we have:

$$7)\qquad \lim_{n=\infty}\frac{1}{\pi}\int_0^\pi f(u+\xi)\,\sigma_n(u)\,du = \frac{1}{2}f(\xi^+).$$

If we replace the condition i) by the requirement that

$$\text{i}')\qquad \lim_{x=\xi^-} f(x) = f(\xi^-);$$

and if, instead of ii), we write:

$$\text{ii}')\qquad \frac{f(\xi+h)-f(\xi^-)}{h}, \qquad \delta < h < 0,$$

demanding that this difference-quotient be bounded, it then follows at once that

$$8)\qquad \lim_{n=\infty}\frac{1}{\pi}\int_{-\pi}^0 f(u+x)\,\sigma_n(u)\,du = \frac{1}{2}f(\xi^-).$$

On combining these two results, assuming that both Conditions i) and i′) are fulfilled, and likewise ii) and ii′), we see that the Fourier's expansion converges at $\xi$ to the value

$$\tfrac{1}{2}[f(\xi^+)+f(\xi^-)].$$

If $f(x)$ is continuous at $\xi$, the series converges to the value $f(\xi)$. We have thus proved the following theorem.

THEOREM. *Let $f(x)$ be continuous in the interval $(-\pi, \pi)$ except at most for a finite number of points, and let $f(x)$ be absolutely integrable:*

$$\int_{-\pi}^{\pi} |f(x)|\,dx \qquad \textit{converges.}$$

*Then the Fourier's series converges toward the value of the function in each point in which $f(x)$ has a derivative.*

*More generally, let the condition of differentiability be replaced by the requirement that the limits*

$$\lim_{x=\xi^+} f(x) = f(\xi^+), \qquad \lim_{x=\xi^-} f(x) = f(\xi^-)$$

*exist and the difference quotients*

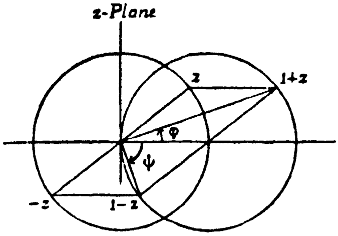
$$\frac{f(\xi+h)-f(\xi^+)}{h}, \quad 0<h<\delta; \quad \frac{f(\xi+h)-f(\xi^-)}{h}, \quad -\delta<h<0$$

*be bounded. Then the Fourier's series converges toward the value:*

$$\tfrac{1}{2}[f(\xi^+)+f(\xi^-)].$$

*Evaluation through Complex Variables.* Consider the following development in the complex domain:

$$1) \qquad \log\frac{1+z}{1-z} = 2\left(z+\frac{z^3}{3}+\frac{z^5}{5}+\cdots\right), \qquad z = re^{\theta i}.$$



The circle of convergence is the unit circle. The series converges in every point of the circumference, except $z = \pm 1$, to the value

$$2\left(\cos\theta+\frac{\cos 3\theta}{3}+\frac{\cos 5\theta}{5}+\cdots\right)$$

$$+\,2i\left(\sin\theta+\frac{\sin 3\theta}{3}+\frac{\sin 5\theta}{5}+\cdots\right).$$

Since the function is continuous there, the value of the function and the value of the series are the same, by Abel's Theorem, Chap. V, § 6. Now, the angle of the argument of the function is

$$\operatorname{arc}(1+z)-\operatorname{arc}(1-z) = \begin{cases} \varphi-\psi = \frac{\pi}{2}, & 0<\theta<\pi; \\ \psi-\varphi = -\frac{\pi}{2}, & -\pi<\theta<0; \end{cases}$$

and this is the coefficient of the pure imaginary part of the logarithm:

$$\log(X + Yi) = \tfrac{1}{2}\log(X^2 + Y^2) + i \operatorname{arc}(X + Yi),$$

or

$$\log[R(\cos\Theta + i\sin\Theta)] = \log R + \Theta i.$$

Hence

$$2)\qquad \frac{\sin\theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots = \begin{cases} \tfrac{1}{4}\pi, & 0 < \theta < \pi; \\ 0, & \theta = 0; \\ -\tfrac{1}{4}\pi, & -\pi < \theta < 0. \end{cases}$$

In conclusion, an appreciation of the rôle which Fourier's series have played in the developement of modern mathematics is found in the retiring address of Professor Edward Burr Van Vleck as Vice-President of Section A, A.A.A.S., published in *Science*, vol. 39, 1914.

## EXERCISES

1. By means of the development:

$$3)\qquad \log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

show that

$$4)\qquad \frac{\theta}{2} = \sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \cdots, \qquad -\pi < \theta < \pi.$$

2. From the series of Question 1 show that

$$5)\qquad \frac{\pi - \theta}{2} = \sin\theta + \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} + \cdots, \qquad 0 < \theta < 2\pi.$$

3. From 4) and 5) deduce 2).

4. Let $f(x)$ be continuous in the interval

$$0 \leqq x \leqq \pi,$$

and let $f(0) = 0$, $f(\pi) = 0$. Let $f(x)$ have a continuous first derivative except at a finite number of points, at each of which it approaches a limit from above, and also a limit from below. Show that the series

$$\sum_{n=1}^{\infty} b_n e^{-ny} \sin nx, \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx,$$

defines a function, $u = F(x, y)$, continuous in the region

$$0 \leqq x \leqq \pi, \qquad 0 \leqq y < \infty,$$

and satisfying Laplace's Equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

at all points $(x, y)$ of the region, for which $0 < y$. Moreover, $u$ takes on the boundary values

$$\text{i)} \qquad F(0, y) = 0;$$

$$\text{ii)} \qquad F(\pi, y) = 0;$$

$$\text{iii)} \qquad F(x, 0) = f(x);$$

and

$$\text{iv)} \qquad \lim_{y=\infty} F(x, y) = 0,$$

no matter how $x$ varies.

5. Consider the region

$R$: $$0 \leqq x \leqq \pi, \qquad 0 \leqq t < \infty.$$

Let $f(x)$ satisfy the same conditions as in Question 4. Show that the series

$$\sum_{n=1}^{\infty} b_n e^{-n^2a^2t} \sin nx, \qquad b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\, dx,$$

defines a function $u = F(x, t)$ which satifies the Heat Equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

at all interior points of $R$ and also in the boundary points

$$x = 0, \quad 0 < t < \infty \qquad \text{and} \qquad x = \pi, \quad 0 < t < \infty.$$

Moreover, $u$ takes on the boundary values:

$$\text{i)} \qquad F(0, t) = 0;$$

$$\text{ii)} \qquad F(\pi, t) = 0;$$

$$\text{iii)} \qquad F(x, 0) = f(x),$$

and

$$\text{iv)} \qquad \lim_{t=\infty} F(x, t) = 0,$$

no matter how $x$ varies.

6. Let it be required to solve the Heat Equation·

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

in the region

$R$: $\qquad 0 \leqq x \leqq \pi, \qquad 0 \leqq t < \infty$:

$$u = \Phi(x, t),$$

subject to the boundary conditions

i) $\Phi(0, t) = 0$;

ii) $\Phi(\pi, t) = c$;

iii) $\Phi(x, 0) = \varphi(x)$;

iv) $\Phi(x, \infty) = 0$,

where $\varphi(x)$ is continuous and $\varphi(0) = 0$, $\varphi(\pi) = c$; moreover, $\varphi(x)$ shall have a continuous derivative except at a finite number of points, at each of which the derivative shall approach a limit from above, and also a limit from below.

Show that this problem is referred to that of Question 5 by setting

$$f(x) = \varphi(x) - \frac{c}{\pi} x,$$

$$\Phi(x, t) = F(x, t) + \frac{c}{\pi} x.$$

# Chapter IX

## Definite Integrals. Line Integrals

**§ 1. Proper Integrals. Continuity.** Consider the integral

$$\int_a^b f(x, \alpha)\, dx.$$

If we impose on the function $f(x, \alpha)$ merely enough conditions to insure the convergence of the integral, the function $\varphi(\alpha)$ represented by the integral

$$\varphi(\alpha) = \int_a^b f(x, \alpha)\, dx,$$

will have no properties; i.e. it may be any function whatever. For, choose $\psi(\alpha)$ arbitrarily, and set

$$f(x, \alpha) = \frac{\psi(\alpha)}{b-a}.$$

Then

$$\varphi(\alpha) = \int_a^b \frac{\psi(\alpha)}{b-a}\, dx = \psi(\alpha).$$

We will begin by restricting the integrand as follows.

THEOREM. *Let the function $f(x, \alpha)$ be continuous in the closed region:*

$R$: $\qquad a \leqq x \leqq b, \qquad A \leqq \alpha \leqq B.$

*Then the function $\varphi(\alpha)$ defined by the integral:*

1) $$\varphi(\alpha) = \int_a^b f(x, \alpha)\, dx,$$

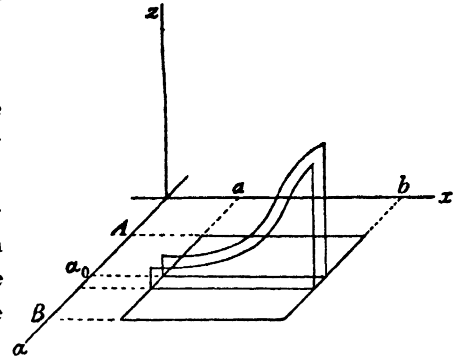
*will be continuous in the closed interval*

$$A \leqq \alpha \leqq B.$$

Geometrically, the truth of the theorem is at once obvious. For, represent the integrand by a surface:

$$2) \qquad z = f(x, \alpha).$$

Cut the surface by the plane $\alpha = \alpha_0$, where $\alpha_0$ is any point of the interval $(A, B)$. Then the value of the integral, or $\varphi(\alpha_0)$, is given by the area under the curve of intersection of the plane with the surface.



Next, cut the surface by the plane $\alpha = \alpha_0 + \Delta\alpha$. The value of the integral, or $\varphi(\alpha_0 + \Delta\alpha)$, is now represented by the area of a near-by curve, and so does not differ much from the former area. Hence $\varphi(\alpha)$ is continuous.

The arithmetic proof is as follows. Since $f(x, \alpha)$ is continuous in the closed region $R$, it is uniformly continuous there. To an arbitrary positive $\epsilon$, then, there corresponds a positive $\delta$, independent of $(x, \alpha)$, $(x', \alpha')$ and such that

$$|f(x, \alpha) - f(x', \alpha')| < \epsilon, \qquad \begin{cases} |x - x'| < \delta \\ |\alpha - \alpha'| < \delta \end{cases}$$

provided $(x, \alpha)$, $(x', \alpha')$ are in $R$. Now form the difference:

$$\varphi(\alpha_0 + \Delta\alpha) - \varphi(\alpha_0) = \int_a^b \{ f(x, \alpha_0 + \Delta\alpha) - f(x, \alpha_0) \} \, dx.$$

Let $\Delta\alpha$ be restricted to the interval $|\Delta\alpha| < \delta$. Then

$$|f(x, \alpha_0 + \Delta\alpha) - f(x, \alpha_0)| < \epsilon,$$

and so

$$|\varphi(\alpha_0 + \Delta\alpha) - \varphi(\alpha_0)| < \int_a^b \epsilon \, dx = (b - a)\epsilon, \qquad |\Delta\alpha| < \delta.$$

This completes the proof.

## EXERCISES

1. Criticise the following proof. "In the equation:

$$\varphi(\alpha_0 + \Delta\alpha) - \varphi(\alpha_0) = \int_a^b \{f(x, \alpha_0 + \Delta\alpha) - f(x, \alpha_0)\}\, dx,$$

let $\Delta\alpha$ approach 0. Since

$$\lim_{\Delta\alpha=0} \{f(x, \alpha_0 + \Delta\alpha) - f(x, \alpha_0)\} = 0$$

the right-hand side approaches 0. Hence the left-hand side approaches 0, and the function $\varphi(\alpha)$ is continuous."

2. Prove that

$$\int_\xi^x f(x, \alpha)\, dx$$

is a continuous function of $(\alpha, x, \xi)$, where $f(x. \alpha)$ satisfies the conditions of the theorem, and $a \leqq x \leqq b$, $a \leqq \xi \leqq b$.

3. By means of the equation

$$\int_1^x x^{\alpha-1}\, dx = \frac{x^\alpha - 1}{\alpha}, \qquad 0 < x,$$

prove that

$$\lim_{(x,\alpha)=(\xi,0)} \frac{x^\alpha - 1}{\alpha} = \log \xi, \qquad 0 < \xi.$$

4. Let $f(x, \alpha)$ be a function which, for each $\alpha$ of an infinite point set $A$, is continuous in the closed interval

$$a \leqq x \leqq b.$$

Let $\alpha_0$ be a point of condensation of $A$, but not necessarily a point of $A$. Let $f(x, \alpha)$ approach a limit when $(x, \alpha)$ approaches $(x_0, \alpha_0)$, where $x_0$ is any point of the interval $(a, b)$. Then the function

$$\varphi(\alpha) = \int_a^b f(x, \alpha)\, dx$$

approaches a limit when $\alpha$ approaches $\alpha_0$.

Moreover, the function

$$\lim_{\alpha=\alpha_0} f(x, \alpha) = \omega(x)$$

is continuous in the interval $a \leqq x \leqq b$, and

$$\lim_{\alpha=\alpha_0} \varphi(\alpha) = \int_a^b \omega(x)\,dx.$$

Succinctly, then,

$$\lim_{\alpha=\alpha_0} \int_a^b f(x, \alpha)\,dx = \int_a^b \lim_{\alpha=\alpha_0} f(x, \alpha)\,dx.$$

§ 2. **Continuation.** *Several Parameters.* The integrand may depend on several parameters : $f(x, \alpha_1, \alpha_2, \cdots, \alpha_n)$. Let $(\alpha_1, \cdots, \alpha_n)$ be any point of a closed region $B$ (Chap. III, § 1) of the $n$-dimensional space of the $\alpha$'s ; and let $x$ lie in the closed interval

$$a \leqq x \leqq b.$$

Let $f(x, \alpha_1, \cdots, \alpha_n)$ be continuous in the closed region thus defined in the $(n+1)$-dimensional space of the $(x, \alpha_1, \cdots, \alpha_n)$. Then

$$1) \qquad \varphi(\alpha_1, \cdots, \alpha_n) = \int_a^b f(x, \alpha_1, \cdots, \alpha_n)\,dx$$

is continuous in $B$. — The proof is essentially the same as in the earlier case.

*Multiple Integrals.* Let $\tau$ be a region of the $(x, y, z)$-space, and let $f(x, y, z, \alpha_1, \cdots, \alpha_n)$ be continuous in the region $R$ of the $(3+n)$-dimensional space defined by $\tau$ and $B$. Then the triple integral of $f$, extended over $\tau$, defines a continuous function of the $\alpha$'s:

$$2) \qquad \varphi(\alpha_1, \cdots, \alpha_n) = \iiint_\tau f(x, y, z, \alpha_1, \cdots, \alpha_n)\,d\tau,$$

where $\varphi(\alpha_1, \cdots, \alpha_n)$ is continuous in $B$.

Of course, a similar remark applies to double and surface integrals on the one hand :

$$3) \qquad \varphi(\alpha_1, \cdots, \alpha_n) = \iint_\sigma f(x, y, \alpha_1, \cdots, \alpha_n)\,d\sigma$$

and to $m$-fold volume or (hyper-) surface integrals on the other :

$$4) \qquad \varphi(\alpha_1, \cdots, \alpha_n) = \underbrace{\int \cdots \int}_{\tau} f(x_1, \cdots, x_m, \alpha_1, \cdots, \alpha_n)\,d\tau;$$

the function $\varphi(\alpha_1, \cdots, \alpha_n)$ being in each case continuous in $B$, and $f$ continuous in $R$.

For example, consider the integrals that define a Newtonian potential function:

$$5)\qquad \iiint\limits_{V} \frac{\rho\, dV}{r}, \qquad \iint\limits_{S} \frac{\sigma\, dS}{r},$$

$V$ and $S$ here denoting the region $\tau$ or $\sigma$, and $\rho$, $\sigma$ meaning the volume or the surface density. Thus

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2,$$

$$\rho = \rho\,(x, y, z), \quad \sigma = \sigma\,(x, y),$$

or, in the case of a curved surface $S$, $\sigma$ is a continuous function on $S$.

*The Iterated Integral.* It is precisely these theorems that are needed to complete the proof of evaluation of the multiple integral by means of the iterated integral[*]. Thus in establishing the evaluation:

$$6)\qquad \iint\limits_{S} f(x, y)\, dS = \int_a^b dx \int_{Y_1}^{Y_2} f(x, y)\, dy,$$

it is essential to know that the first integral,

$$7)\qquad \int_{Y_1}^{Y_2} f(x, y)\, dy,$$

is a continuous function of $x$, and this brings us to the last of the generalizations, namely:

THEOREM 2. *Let $f(x, \alpha)$ be continuous in the closed region* $R$: $a \leqq x \leqq b$, $A \leqq \alpha \leqq B$, *where*

$$a = \psi(\alpha), \qquad b = \omega(\alpha), \qquad \psi(\alpha) < \omega(\alpha).$$

*and $\psi(\alpha)$, $\omega(\alpha)$ are any functions continuous in the interval $A \leqq \alpha \leqq B$. Then the integral from $a$ to $b$, of the function $f$, is a continuous function of $\alpha$:*

$$8)\qquad \varphi(\alpha) = \int_a^b f(x, \alpha)\, dx.$$

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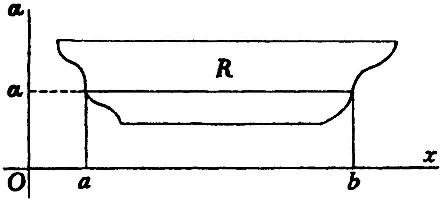
[*] cf. The Author's *Advanced Calculus*, p. 260, and the *Funktionentheorie*, vol. I, 1928, p. 118.

Here, again, the intuitional proof by geometry — the area under the surface, § 1 — is suggestive and convincing — as convincing as intuitional geometry can be.



The analytic proof is simple. Make a change of variable:

$$t = \frac{x-a}{b-a}.$$

Then

$$9) \qquad \int_a^b f(x, \alpha)\,dx = \int_0^1 (b-a) f[(b-a)t + a, \alpha]\,dt.$$

The latter integral represents a continuous function by Theorem 1, § 1.

The extension to the case of $n$ parameters, $(\alpha_1, \cdots, \alpha_n)$, considered in a region $B$, is immediate, the proof requiring no modification:

$$10) \qquad \varphi(\alpha_1, \cdots, \alpha_n) = \int_a^b f(x, \alpha_1, \cdots, \alpha_n)\,dx,$$

$$a = \psi(\alpha_1, \cdots, \alpha_n),\ b = \omega(\alpha_1, \cdots, \alpha_n),\ \psi(\alpha_1, \cdots, \alpha_n) < \omega(\alpha_1, \cdots, \alpha_n).$$

## § 3. Differentiation. Leibniz's Rule.

Consider the differentiation of the function

$$1) \qquad \varphi(\alpha) = \int_a^b f(x, \alpha)\,dx.$$

Form the difference-quotient:

$$\frac{\varphi(\alpha_0 + \Delta\alpha) - \varphi(\alpha_0)}{\Delta\alpha} = \int_a^b \frac{f(x, \alpha_0 + \Delta\alpha) - f(x, \alpha_0)}{\Delta\alpha}\,dx.$$

If the partial derivative of $f(x, \alpha)$ with respect to $\alpha$ exists:

$$\lim_{\Delta\alpha \doteq 0} \frac{f(x, \alpha_0 + \Delta\alpha) - f(x, \alpha_0)}{\Delta\alpha} = \frac{\partial f}{\partial \alpha},$$

the integrand approaches a limit when $\Delta\alpha$ approaches 0, and it was formerly considered self-evident that

$$2)\qquad \frac{d\varphi}{d\alpha}=\int_a^b \frac{\partial f}{\partial\alpha}\,dx,$$

or that the definite integral can be differentiated under the sign of integration. The formula 2) is known as *Leibniz's Rule.*

Here is, of course, a double-limit fallacy. What we want is:

$$\lim_{\Delta\alpha=0}\frac{\Delta\varphi}{\Delta\alpha},\qquad\text{or}\qquad \lim_{\Delta\alpha=0}\int_a^b \frac{f(x,\alpha_0+\Delta\alpha)-f(x,\alpha_0)}{\Delta\alpha}\,dx,$$

and what we have found is:

$$\int_a^b \lim_{\Delta\alpha=0}\frac{f(x,\alpha_0+\Delta\alpha)-f(x,\alpha_0)}{\Delta\alpha}\,dx,\qquad\text{or}\qquad \int_a^b\frac{\partial f}{\partial\alpha}\,dx.$$

Nevertheless, under suitable restrictions, both limits exist and the two are equal.

THEOREM. *Let $f(x,\alpha)$ be a continuous function of $x$ in the closed interval $a \leqq x \leqq b$, $\alpha$ having any fixed value in the interval $A \leqq \alpha \leqq B$. Let $\partial f/\partial\alpha$ exist at each point of the region*

$$R:\qquad a \leqq x \leqq b,\qquad A \leqq \alpha \leqq B,$$

*and let the function*

$$\frac{\partial f}{\partial\alpha}=f_\alpha(x,\alpha)$$

*be continuous in $R$. Let*

$$\varphi(\alpha)=\int_a^b f(x,\alpha)\,dx.$$

*Then $\varphi(\alpha)$ has a derivative, given by Leibniz's Rule:*

$$\frac{d\varphi}{d\alpha}=\int_a^b\frac{\partial f}{\partial\alpha}\,dx,$$

*and $\varphi'(\alpha)$ is continuous, $A \leqq \alpha \leqq B$.*

By the Law of the Mean

$$f(x, \alpha_0 + \Delta\alpha) - f(x, \alpha_0) = \Delta\alpha \, f_\alpha(x, \alpha_0 + \theta\Delta\alpha), \qquad 0 < \theta < 1.$$

The function $f_\alpha(x, \alpha)$ is uniformly continuous in $R$. Hence to an arbitrary $\epsilon > 0$ corresponds a $\delta > 0$, independent of $(x, \alpha)$, $(x', \alpha')$, such that

$$|f_\alpha(x, \alpha) - f_\alpha(x', \alpha')| < \epsilon, \qquad \begin{cases} |x - x'| < \delta \\ |\alpha - \alpha'| < \delta \end{cases}$$

It follows, then, that

$$\frac{\varphi(\alpha_0 + \Delta\alpha) - \varphi(\alpha_0)}{\Delta\alpha} = \int_a^b f_\alpha(x, \alpha_0 + \theta\,\Delta\alpha)\,dx,$$

$$\left| \frac{\varphi(\alpha_0 + \Delta\alpha) - \varphi(\alpha_0)}{\Delta\alpha} - \int_a^b f_\alpha(x, \alpha_0)\,dx \right| \leqq$$

$$\int_a^b |f_\alpha(x, \alpha_0 + \theta\,\Delta\alpha) - f_\alpha(x, \alpha_0)|\,dx < \int_a^b \epsilon\,dx = (b - a)\,\epsilon,$$

provided $|\Delta\alpha| < \delta$. This proves the theorem.

The extension to the case of several parameters and multiple integrals is immediate. Let $\tau$ be a closed region of the $(x_1, \cdots, x_m)$-space; $B$, a region of the $(\alpha_1, \cdots, \alpha_n)$-space, and let $f(x_1, \cdots, x_m, \alpha_1, \cdots, \alpha_n)$ be defined in every point of the region $R$ of the $(x_1, \cdots, x_m, \alpha_1, \cdots, \alpha_n)$-space determined by $\tau$ and $B$. For an arbitrary point $(\alpha)$ in $R$ let $f$ be continuous in $\tau$. Then the integral of $f$, extended throughout $\tau$, defines a function $\varphi$ of the $(\alpha)$:

$$\varphi(\alpha_1, \cdots, \alpha_n) = \underbrace{\int \cdots \int}_{\tau} f(x_1, \cdots, x_m, \alpha_1, \cdots, \alpha_n)\,d\tau,$$

Let

$$\frac{\partial f}{\partial \alpha_k} = f_k(x_1, \cdots, x_m, \alpha_1, \cdots, \alpha_n)$$

exist and be continuous in the interior points of the $(m + n)$-dimensional region $R$, and bounded. Then $\varphi$ admits a partial derivative $\partial\varphi/\partial\alpha_k$, given by Leibniz's Rule:

$$\frac{\partial \varphi}{\partial \alpha_k} = \underbrace{\int \cdots \int}_{\tau} \frac{\partial f}{\partial \alpha_k}\,d\tau,$$

and the function

$$\frac{\partial \varphi}{\partial \alpha_k} = \varphi_k(\alpha_1, \cdots, \alpha_n)$$

is continuous in $B$.

*Example.* Consider the potential:

$$u = \iiint\limits_{\tau} \frac{\rho\, d\tau}{r},$$

§ 2. Hence,

$$\frac{\partial u}{\partial a} = \iiint\limits_{\tau} \frac{\rho\,(x-a)\, d\tau}{r^3}, \qquad \text{etc.}$$

Further, since $1/r$ is a solution of Laplace's equation,

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

it follows that $u$ is, also. For

$$\Delta u = \iiint\limits_{\tau} \rho\, \Delta\left(\frac{1}{r}\right) d\tau = 0.$$

### EXERCISE

Differentiate the integral:

$$\int_0^1 \frac{dx}{1 + x + \alpha^2}$$

by Leibniz's Rule, and verify the result by direct computation.

**§ 4. Variable Limits of Integration.** THEOREM. *Let $f(x, \alpha)$ be defined in the region*

$R$: $\qquad a \leqq x \leqq b, \qquad A \leqq \alpha \leqq B,$

$$a = \psi(\alpha), \qquad b = \omega(\alpha), \qquad \psi(\alpha) < \omega(\alpha),$$

*where $\psi(\alpha)$, $\omega(\alpha)$ have continuous derivatives in the interval $A \leqq \alpha \leqq B$. Let $f(x, \alpha)$ be continuous in $x$ for each value of $\alpha$: $a \leqq x \leqq b$. Finally, let*

$$\frac{\partial f}{\partial \alpha} = f_\alpha(x, \alpha)$$

*exist at every point of R not on the boundary* $a = \psi(\alpha)$, $b = \omega(\alpha)$, *and let* $f_\alpha(x, \alpha)$ *be continuous on the boundary. Then the function*

$$\varphi(\alpha) = \int_a^b f(x, \alpha)\, dx$$

*admits a derivative, continuous in the interval* $A \leqq \alpha \leqq B$, *and given by the formula:*

$$\frac{d\varphi}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}\, dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha}.$$

The proof is simple in case the definition of the function can be extended to a somewhat larger region,

$$R': \qquad a \leqq x \leqq b, \qquad A \leqq \alpha \leqq B,$$

$$a = \psi_1(\alpha) < \psi(\alpha), \qquad b = \omega_1(\alpha) > \omega(\alpha).$$

For then we have:

$$\varphi(\alpha_0 + \Delta\alpha) - \varphi(\alpha_0) = \int_{a_0}^{b_0} \{f(x, \alpha_0 + \Delta\alpha) - f(x, \alpha_0)\}\, dx$$

$$+ \int_{b_0}^{b_0+\Delta b} f(x, \alpha_0 + \Delta\alpha)\, dx + \int_{a_0+\Delta a}^{a_0} f(x, \alpha_0 + \Delta\alpha)\, dx.$$

The first integral can be treated as in § 3. The second can be appraised by the Law of the Mean for integrals:

$$\int_{b_0}^{b_0+\Delta b} f(x, \alpha_0 + \Delta\alpha)\, dx = \Delta b f(X, \alpha_0 + \Delta\alpha),$$

where $X$ lies between $b_0$ and $b_0 + \Delta b$. And the last integral can be represented in a similar manner. Thus

$$\frac{\Delta\varphi}{\Delta\alpha} = \int_{a_0}^{b_0} f_\alpha(x, \alpha_0 + \theta\,\Delta\alpha)\, dx +$$

$$\frac{\Delta b}{\Delta\alpha} f(X, \alpha_0 + \Delta\alpha) - \frac{\Delta a}{\Delta\alpha} f(X', \alpha_0 + \Delta\alpha),$$

$$\lim_{\Delta\alpha=0} X = b_0, \qquad \lim_{\Delta\alpha=0} X' = a_0,$$

and the completion of the proof is now easy.

When such an extension of the definition is not obvious or possible, the above proof can be modified without great difficulty, but there is a simpler treatment, which we will not undertake to reproduce here; cf. the Author's *Funktionentheorie*, vol. I, 1928, p. 122. On the other hand, if one adds to the restrictions on $f(x, \alpha)$ the requirement of the existence and continuity of $\partial f/\partial x$, the transformation of the variable of integration used in § 2, whereby

$$\int_a^b f(x, \alpha)\, dx = \int_0^1 (b - a) f[(b - a)t + a, \alpha]\, dt,$$

leads at once to a proof.

## § 5. Iterated Integral with Constant Limits.

THEOREM. *Let $f(x, y)$ be continuous in the rectangle*

$R$: $\qquad a \leqq x \leqq b, \qquad A \leqq y \leqq B.$

*Then*

$$\int_a^b dx \int_A^B f(x, y)\, dy = \int_A^B dy \int_a^b f(x, y)\, dx.$$

It is possible to give a simple proof of this theorem without recourse to the double integral. Form the function

$$F(x, y) = \int_a^x dx \int_A^y f(x, y)\, dy.$$

The function

$$\int_A^y f(x, y)\, dy$$

is continuous in $R$, cf. § 1, Exercise 2, and it has a continuous derivative with respect to $y$. Hence by § 3

$$\frac{\partial F}{\partial x} = \int_A^y f(x, y)\, dy, \qquad \frac{\partial F}{\partial y} = \int_a^x f(x, y)\, dx.$$

Thus

$$\int_A^B f(x,y)\,dy = F_x(x,B) - F_x(x,A)$$

and

$$\int_a^b dx \int_A^B f(x,y)\,dy =$$

$$F(b,B) - F(a,B) - F(b,A) + F(a,A).$$

Because of the symmetry in the result it is obvious that the iterated integral taken in the reverse order has the same value, and this completes the proof.

**§ 6. Proof that** $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$. Let $u = F(x,y)$ be continuous in the neighborhood of a point $(x_0, y_0)$. Let the first partial derivatives exist and be continuous in this region, and also the second partial derivatives in question. Choose a rectangle $R$, § 5, containing the point $(x_0, y_0)$ in its interior and itself lying wholly within the above neighborhood. Let

1) $$\Phi(x,y) = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x}$$

and compute the integral:

2) $$\int_\xi^x dx \int_\eta^y \Phi(x,y)\,dx \qquad \begin{cases} a \leqq \xi < x \leqq b \\ A \leqq \eta < y \leqq B \end{cases}$$

Its value by § 5 is:

3) $$\int_\xi^x dx \int_\eta^y \frac{\partial^2 u}{\partial x \partial y}\,dy - \int_\eta^y dy \int_\xi^x \frac{\partial^2 u}{\partial y \partial x}\,dx.$$

Each of these integrals has the value:

$$F(x,y) - F(\xi,y) - F(x,\eta) + F(\xi,\eta).$$

Consequently the integral 2) has the value 0, and hence the function $\Phi(x,y)$ vanishes identically. For, if $\Phi(x,y)$ were positive (negative) at a point within $R$, the limits of integration in 2) could be so chosen that the integral 2) would be positive (negative). It follows, then, that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

**§ 7. Improper Integrals.**[*] Consider the improper integral

$$1) \qquad \int_c^\infty f(x, \alpha)\, dx.$$

The integrand shall be continuous in the region

$$R: \qquad c \leqq x < \infty, \qquad A \leqq \alpha \leqq B,$$

and the integral shall converge for all values of $\alpha$ in the interval $(A, B)$. Thus the integral defines a function of $\alpha$:

$$2) \qquad \varphi(\alpha) = \int_c^\infty f(x, \alpha)\, dx.$$

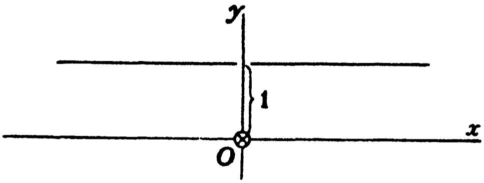
The function $\varphi(\alpha)$ is not, however, in general continuous. For example, the integral

$$3) \qquad \int_0^\infty \alpha^2 e^{-\alpha^2 x}\, dx$$

converges for all values of $\alpha$. Its value, $\varphi(\alpha)$, is 1 when $\alpha \neq 0$; but when $\alpha = 0$, it vanishes:

$$\varphi(\alpha) = 1, \qquad \alpha \neq 0; \qquad \varphi(0) = 0.$$

Its graph, $y = \varphi(\alpha)$, is a line parallel to the axis of $\alpha$ and 1 unit above it, except when $\alpha = 0$; then it drops to the origin.



Again, consider the integral

$$4) \qquad \psi(\alpha) = \int_0^\infty \alpha^3 e^{-\alpha^2 x}\, dx.$$

This integral converges for all values of $\alpha$, and

$$\psi(\alpha) = \alpha.$$

---

[*] The following treatment presupposes the ordinary tests for convergence as developed in the Calculus; cf. for example, the Author's *Advanced Calculus*, Cap. XIX.

Moreover, $\partial f/\partial\alpha$ exists and is continuous for all points $0 \leqq x < \infty$, $-\infty < \alpha < \infty$:

$$\frac{\partial f}{\partial\alpha} = (3\alpha^2 - 2\alpha^4 x)\, e^{-\alpha^2 x}.$$

Nevertheless, differentiation under the sign of integration is impossible when $\alpha = 0$. For

$$\frac{d\psi}{d\alpha} = 1,$$

but the integral

$$\int_0^\infty \frac{\partial f}{\partial\alpha}\, dx = 0$$

when $\alpha = 0$.

And still again, it is not true that

$$\int_{\alpha_0}^{\alpha_1} d\alpha \int_c^\infty f(x,\alpha)\, dx = \int_c^\infty dx \int_{\alpha_0}^{\alpha_1} f(x,\alpha)\, d\alpha,$$

even when $f(x.\alpha)$ satisfies the above requirements and all the integrals involved converge. For example, let

$$f(x,\alpha) = (2\alpha - 2\alpha^3 x)\, e^{-\alpha^2 x}.$$

It is readily shown by direct computation that

$$\int_0^\infty f(x,\alpha)\, dx = 0$$

for all values of $\alpha$. Hence

$$\int_0^\alpha d\alpha \int_0^\infty f(x,\alpha)\, dx = 0.$$

On the other hand,

$$\int_0^\alpha f(x,\alpha)\, d\alpha = \alpha^2 e^{-\alpha^2 x}.$$

Thus

$$\int_0^\infty dx \int_0^\alpha f(x,\alpha)\, d\alpha = \int_0^\infty \alpha^2 e^{-\alpha^2 x}\, dx.$$

This is the integral 3), and its value is 1 when $\alpha \neq 0$.

**§ 8. Double Limits.** The phenomena described in the last paragraph are all examples of *double limits*:

$$\lim_{\alpha=\alpha_0}\int_c^\infty f(x,\alpha)\,dx \qquad \text{and} \qquad \int_c^\infty \lim_{\alpha=\alpha_0} f(x,\alpha)\,dx\,;$$

$$\lim_{\Delta\alpha=0}\frac{\Delta\int_c^\infty f(x,\alpha)\,dx}{\Delta\alpha} \qquad \text{and} \qquad \int_c^\infty \lim_{\Delta\alpha=0}\frac{\Delta f}{\Delta\alpha}\,dx;$$

$$\lim_{n=\infty}\sum_{k=1}^{n}\Delta\alpha_k\int_c^\infty f(x,\alpha_k')\,dx \qquad \text{and} \qquad \int_c^\infty \lim_{n=\infty}\sum_{k=1}^{n} f(x,\alpha_k')\,dx.$$

The questions are precisely similar to those that arose in infinite series:

i) Continuity of the function defined by a convergent series of continuous functions;

ii) Differentiation of a series term-by-term;

iii) Integration of a series term-by-term.

In the earlier case we found that *uniform convergence* was the key to the situation, for it enabled us each time to infer the equality of the two double limits. The same holds true here. We lay down a definition of uniform convergence for an integral analogous to the definition of uniform convergence for a series, and apply it in a similar manner.

In a larger sense both definitions of uniform convergence come under the general case embodied in the theorem of Chap. V, § 10.

**§ 9. Uniform Convergence.** *Definition.* Let $f(x,\alpha)$ be a function which, for each point $\alpha$ of an infinite point set $A$, is continuous in $x$,

$$c \leqq x < \infty.$$

The integral

$$\int_c^\infty f(x,\alpha)\,dx \tag{1}$$

is said to *converge uniformly in the point set $A$* if, to a positive $\epsilon$

chosen at pleasure there, corresponds a number $g$ ($\geqq c$) *independent of* $\alpha$ and such that

$$\left|\int_{x'}^{x''} f(x, \alpha)\, dx\right| < \epsilon, \qquad g \leqq x', x''.$$

THEOREM I. *A necessary and sufficient condition for the uniform convergence of the integral* 1) *is that, to a positive* $\epsilon$ *chosen at pleasure, there correspond a number* $g \geqq c$, *independent of* $\alpha$ *and such that*

$$\left|\int_{g}^{x} f(x, \alpha)\, dx\right| < \epsilon, \qquad g \leqq x.$$

THEOREM II. *A necessary and sufficient* condition for the uniform convergence of the integral* 1) *is that, to a positive* $\epsilon$ *chosen at pleasure, there correspond a number* $g$ ($\geqq c$) *independent of* $\alpha$ *and such that*

$$\left|\int_{x}^{\infty} f(x, \alpha)\, dx\right| < \epsilon, \qquad g \leqq x < \infty$$

**§ 10. The de la Vallée-Poussin $\mu(x)$ - Test.** Weierstrass's *M*-Test for uniform convergence in the case of infinite series finds its exact counterpart in de la Vallée-Poussin's $\mu(x)$ - Test in the case of definite integrals.

DE LA VALLEE-POUSSIN'S $\mu(x)$-TEST. *Let* $\mu(x)$ *be a function continuous for* $c \leqq \gamma \leqq x$. *If*

i) $$|f(x, \alpha)| \leqq \mu(x), \qquad \gamma \leqq x, \quad A \leqq \alpha \leqq B;$$

ii) $$\int_{\gamma}^{\infty} \mu(x)\, dx$$

*converges, then the integral*

$$\int_{c}^{\infty} f(x, \alpha)\, dx$$

*converges uniformly in the interval* $A \leqq \alpha \leqq B$.

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* For the sufficient condition one must, of course, begin by requiring the plain convergence of the integral 1)

*Proof.* Let $\epsilon > 0$ be chosen arbitrarily. Then $g$ ($\geqq \gamma$) can be so determined that

$$\int_{x'}^{x''} \mu(x)\,dx < \epsilon, \qquad g \leqq x' < x''.$$

Now,

$$\left|\int_{x'}^{x''} f(x,\alpha)\,dx\right| \leqq \int_{x'}^{x''} |f(x,\alpha)|\,dx \leqq \int_{x'}^{x''} \mu(x)\,dx.$$

Hence

$$\left|\int_{x'}^{x''} f(x,\alpha)\,dx\right| < \epsilon, \qquad g \leqq x' < x''.$$

But $g$ is independent of $\alpha$, and so the theorem is proved.

*Example.* The integral

$$\int_{1}^{\infty} x^{\alpha-1} e^{-x}\,dx$$

converges uniformly in any interval bounded from above:

$$\alpha \leqq G.$$

For, let

$$\mu(x) = x^{G-1} e^{-x}.$$

## EXERCISES

1. Show that the integral

$$\int_{1}^{\infty} x^{\alpha-1} e^{-x} \log x\,dx$$

converges uniformly in every interval bounded from above.

2. The same for

$$\int_{2}^{\infty} x^{\alpha-1} e^{-x} (\log x)^{\beta}\,dx.$$

3. What can you say regarding the uniform convergence of the integral

$$\int_{0}^{\infty} e^{-x^2} \cos \alpha x\,dx\;?$$

4. The same for

$$\int_0^\infty \frac{\alpha\, dx}{\alpha^2 + x^2}\,.$$

5. The same for

$$\int_0^\infty x e^{-x^2(1+\alpha^2)}\, dx.$$

**§ 11. Continuity.** THEOREM. *If $f(x, \alpha)$ is continuous in the region*

$R:\qquad c \leqq x < \infty, \qquad A \leqq \alpha \leqq B,$

*and if the integral*

$$\int_c^\infty f(x, \alpha)\, dx$$

*converges uniformly in the interval $A \leqq \alpha \leqq B$, then the integral defines a function,*

$$\varphi(\alpha) = \int_c^\infty f(x, \alpha)\, dx,$$

*continuous in that interval.*

*Proof.* We wish to show that, to an arbitrary $A \leqq \alpha_0 \leqq B$ and to a positive $\epsilon$ chosen at pleasure, there corresponds a positive $\delta$ such that

$$|\varphi(\alpha_0 + h) - \varphi(\alpha_0)| < \epsilon, \qquad |h| < \delta, \qquad A \leqq \alpha_0 + h \leqq B.$$

Now,

$$\varphi(\alpha_0 + h) - \varphi(\alpha_0) = \int_c^g \{f(x, \alpha_0 + h) - f(x, \alpha_0)\}\, dx +$$

$$\int_g^\infty f(x, \alpha_0 + h)\, dx - \int_g^\infty f(x, \alpha_0)\, dx,$$

no matter how $c \leqq g$ be chosen. Let $\epsilon'$ be an arbitrary positive number. Then, by Theorem II, § 9, a number $g \geqq c$ and independent of $\alpha_0$, $h$ can be found such that

$$\left|\int_g^\infty f(x,\alpha_0+h)\,dx\right| < \epsilon', \qquad \left|\int_g^\infty f(x,\alpha_0)\,dx\right| < \epsilon'.$$

Hold this $g$ fast. Then, since

$$\int_c^g f(x,\alpha)\,dx$$

represents a continuous function of $\alpha$ in the interval $A \leqq \alpha \leqq B$ by § 1, it follows that

$$\left|\int_c^g \{f(x,\alpha_0+h)-f(x,\alpha_0)\}\,dx\right| < \epsilon', \qquad |h| < \delta.$$

Hence

$$|\varphi(\alpha_0+h)-\varphi(\alpha_0)| < 3\,\epsilon', \qquad |h| < \delta,$$

and it remains only to choose $\epsilon'$ so that $3\,\epsilon' = \epsilon$.

*Example.* The integral

$$\int_1^\infty x^{\alpha-1}e^{-x}\,dx$$

represents a continuous function for all values of $\alpha$. For, let $\alpha_0$ be an arbitrary value. Let $G$ be chosen $> \alpha_0$. Then the integral converges uniformly in the interval $\alpha \leqq G$ (cf. § 10) and so represents a function continuous at $\alpha = \alpha_0$.

## EXERCISES

1. Let a point set $A = \{\alpha\}$ be given, with a point of condensation, $\alpha_0$. Let $f(x,\alpha)$ be a continuous function of $x$ in the interval

$$c \leqq x < \infty$$

for each $\alpha$, $c$ being a constant.

i) Let $f(x,\alpha)$ converge uniformly in any finite interval,

$$c \leqq x \leqq G,$$

when $\alpha$ approaches $\alpha_0$;

ii) Let

$$\int_c^\infty f(x,\alpha)\,dx$$

converge uniformly in $A$. Then:—

a) $$\lim_{\alpha=\alpha_0} f(x, \alpha) = \omega(x)$$

is continuous, $c \leqq x < \infty$;

b) $$\int_c^\infty \omega(x)\,dx \qquad \text{converges};$$

c) $$\lim_{\alpha=\alpha_0} \int_c^\infty f(x, \alpha)\,dx = \int_c^\infty \lim_{\alpha=\alpha_0} f(x, \alpha)\,dx.$$

2. Let $f(x, \alpha)$ be a function which, for each $\alpha$ of an infinite point set $A$, is continuous in $x$,

$$c \leqq x < \infty.$$

Let $\alpha_0$ be a point of condensation of $A$. For an arbitrary value $x'$ of $x$ let $f(x, \alpha)$ approach a limit:

$$\lim_{(x, \alpha)=(x', \alpha_0)} f(x, \alpha) \qquad \text{exists}, \qquad c \leqq x' < \infty.$$

Finally, let the integral

$$\int_c^\infty f(x, \alpha)\,dx$$

converge uniformly in $A$. Then:—

a) $$\lim_{\alpha=\alpha_0} f(x, \alpha) = \omega(x)$$

is continuous, $c \leqq x < \infty$;

b) $$\int_c^\infty \omega(x)\,dx \qquad \text{converges};$$

c) $$\lim_{\alpha=\alpha_0} \int_c^\infty f(x, \alpha)\,dx = \int_c^\infty \lim_{\alpha=\alpha_0} f(x, \alpha)\,dx.$$

**§ 12. Integration. Reversal of the Order.** THEOREM. *If the function $f(x, \alpha)$ is continuous in the region*

$R$: $\qquad c \leqq x < \infty, \qquad A \leqq \alpha \leqq B,$

*and if the integral* 1), § 9, *converges uniformly in the interval* $A \leqq \alpha \leqq B$:

$$\varphi(\alpha) = \int_c^\infty f(x, \alpha)\, dx,$$

*then*

$$\int_{\alpha_0}^{\alpha_1} \varphi(\alpha)\, d\alpha = \int_c^\infty dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha)\, d\alpha,$$

*or:*

$$\int_{\alpha_0}^{\alpha_1} d\alpha \int_c^\infty f(x, \alpha)\, dx = \int_c^\infty dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha)\, d\alpha.$$

*Proof.* The function $\varphi(\alpha)$ is continuous by § 11. We wish to show that, to an arbitrary $\epsilon > 0$ there corresponds a number $g$ ($\geqq c$) such that

$$\left| \int_{\alpha_0}^{\alpha_1} \varphi(\alpha)\, d\alpha - \int_c^x dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha)\, d\alpha \right| < \epsilon, \qquad g \leqq x.$$

Now,

$$\int_c^\infty f(x, \alpha)\, dx = \int_c^x f(x, \alpha)\, dx + \int_x^\infty f(x, \alpha)\, dx,$$

and for a fixed $x$ each integral on the right is a continuous function of $\alpha$. Hence

$$\int_{\alpha_0}^{\alpha_1} d\alpha \int_c^\infty f(x, \alpha)\, dx = \int_{\alpha_0}^{\alpha_1} d\alpha \int_c^x f(x, \alpha)\, dx + \int_{\alpha_0}^{\alpha_1} d\alpha \int_x^\infty f(x, \alpha)\, dx.$$

The order of integration in the first integral on the right can be reversed, § 5. Thus

$$\int_{\alpha_0}^{\alpha_1} d\alpha \int_c^\infty f(x, \alpha)\, dx - \int_c^x dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha)\, d\alpha = \int_{\alpha_0}^{\alpha_1} d\alpha \int_x^\infty f(x, \alpha)\, dx.$$

Because of the uniform convergence of the given integral,

$$\left| \int_x f(x, \alpha)\, dx \right| < \epsilon', \qquad g \leqq x.$$

Hence

$$\left|\int_{\alpha_0}^{\alpha_1} d\alpha \int_x^\infty f(x, \alpha)\, dx\right| < |\alpha_1 - \alpha_0|\, \epsilon', \qquad g \leqq x,$$

and it remains merely to choose $\epsilon'$ so that $|\alpha_1 - \alpha_0|\, \epsilon' \leqq \epsilon$.

### EXERCISE

Under the conditions of the theorem, show that the integral:

$$\int_c^\infty dx \int_A^\alpha f(x, \alpha)\, d\alpha$$

converges uniformly in the interval $A \leqq \alpha \leqq B$.

### § 13. Leibniz's Rule.

THEOREM. *Let the integral* 1), § 9, *converge in the interval* $A \leqq \alpha \leqq B$*:*

$$\varphi(\alpha) = \int_c^\infty f(x, \alpha)\, dx,$$

*Let* $\partial f / \partial \alpha$ *exist and let the function*

$$\frac{\partial f}{\partial \alpha} = f_\alpha(x, \alpha)$$

*be continuous in the region*

$R$: $\qquad c \leqq x < \infty, \qquad A \leqq \alpha \leqq B.$

*Finally, let the integral*

$$\int_c^\infty f_\alpha(x, \alpha)\, dx$$

*converge uniformly in the interval* $A \leqq \alpha \leqq B$.

*Then the function* $\varphi(\alpha)$ *has a derivative, the derivative is continuous, and it is given by the last integral:*

$$\frac{d\varphi}{d\alpha} = \int_c^\infty f_\alpha(x, \alpha)\, dx.$$

*Proof.* Let

$$\psi(\alpha) = \int_c^\infty f_\alpha(x, \alpha)\, dx.$$

Then

$$\int_A^\alpha \psi(\alpha)\, dx = \int_c^\infty dx \int_A^\alpha \frac{\partial f}{\partial \alpha}\, d\alpha$$

$$= \int_c^\infty \{f(x, \alpha) - f(x, A)\}\, dx$$

$$= \int_c^\infty f(x, \alpha)\, dx - \int_c^\infty f(x, A)\, dx.$$

Hence

$$\int_A^\alpha \psi(\alpha)\, d\alpha = \varphi(\alpha) - \varphi(A),$$

and the proof is now given by the theorem of Chap. IV, § 7.

*Example.* The function defined by

$$\int_1^\infty x^{x-1} e^{-x}\, dx$$

can be differentiated with respect to $\alpha$ for all values of the argument.

### EXERCISES

1. Knowing that

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2},$$

show that

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}, \qquad 0 < \alpha.$$

Hence show that

$$\int_0^\infty x^2 e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}, \qquad 0 < \alpha.$$

2. Obtain a formula for

$$\int_0^\infty x^{2n} e^{-x^2}\, dx, \qquad n = 1, 2, 3, \cdots.$$

**§14. Applications.** It is now possible to complete the proof of certain evaluations of definite integrals studied in the Calculus[*].

*Example* 1. Consider the integral

$$1) \qquad u = \int_0^\infty e^{-x^2} \cos \alpha x\, dx.$$

The convergence for all values of $\alpha$ was shown in the Calculus. Moreover

$$\frac{\partial f}{\partial \alpha} = -x e^{-x^2} \sin \alpha x$$

is continuous in the region

$$R: \qquad 0 \leqq x, \qquad -\infty < \alpha < \infty,$$

and the integral

$$\int_0^\infty - x e^{-x^2} \sin \alpha x\, dx$$

converges uniformly for all values of $\alpha$. Hence Leibniz's Rule is justified. Integration by parts requires only elementary methods. Thus it is seen that

$$\frac{du}{d\alpha} = -\frac{\pi}{2} u.$$

The integral of this equation is the function (Chap. XII, § 7):

$$u = k e^{-\frac{\alpha^2}{4}}.$$

When $\alpha = 0$,

$$u = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Hence the integral 1) has the value:

$$\int_0^\infty e^{-x^2} \cos \alpha x\, dx = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}.$$

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[*] Cf. the Author's *Advanced Calculus*, p. 487.

*Example* 2. The integral

2) $$u = \int_0^\infty e^{-x^2 - \frac{\alpha^2}{x^2}}\, dx$$

can be treated in a similar manner. First,

$$\frac{du}{d\alpha} = -2\int_0^\infty \alpha x^{-2} e^{-x^2-\frac{\alpha^2}{x^2}}\, dx, \qquad \alpha \neq 0.$$

For, this last integral converges uniformly in the interval

$$0 < \delta \leqq \alpha,$$

since

$$\alpha x^{-2} e^{-\frac{\alpha^2}{x^2}} = \frac{1}{\alpha}\, \xi\, e^{-\xi} < \frac{1}{\alpha}, \qquad \xi = \frac{\alpha^2}{x^2},$$

and hence

$$\alpha x^{-2} e^{-x^2-\frac{\alpha^2}{x^2}} \leqq \frac{1}{\delta}\, e^{-x^2} = \mu\,(x).$$

On changing the variable of integration, setting $y = \alpha/x$, it is found that

$$\frac{du}{d\alpha} = -2u.$$

Hence

$$u = Ce^{-2\alpha}, \qquad 0 < \alpha,$$

Let $\alpha$ approach 0. The integral 2) is uniformly convergent for all values of $\alpha$. Hence

$$\lim_{\alpha=0+} u = \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2},$$

and so

$$C = \frac{\sqrt{\pi}}{2}.$$

If $\alpha < 0$, the value of $u$ is the same as for $|\,\alpha\,|$. Hence finally

$$u = \frac{\sqrt{\pi}}{2}\, e^{-2|\alpha|}, \qquad \alpha, \text{ unrestricted.}$$

The function is continuous at the origin (as elsewhere), but it has no derivative at the origin. It has a forward derivative, equal to $-\sqrt{\pi}$; and a backward derivative, $+\sqrt{\pi}$. The function is an even function, and so its graph is symmetric in the axis of ordinates.

### Exercise

Evaluate the integral:

$$\int_0^{\infty} e^{-x^2} \sin \alpha x \, dx.$$

Compute its value for $\alpha = 1$.

**§ 15. The Gamma Function.** Let the Gamma Function be defined by the integral,

$$\Gamma(\alpha) = \int_0 x^{\alpha-1} e^{-x} \, dx. \tag{1}$$

This integral is shown by the ordinary tests to converge for all positive values of $\alpha$: $0 < \alpha$; cf. *Advanced Calculus*, p. 480. We now proceed to show that $\Gamma(\alpha)$ is continuous for all such values of $\alpha$.

Let the integral be written as:

$$\int_0^{\infty} = \int_0^{1} + \int_1^{\infty}$$

The second integral,

$$\psi(\alpha) = \int_1^{\infty} x^{\alpha-1} e^{-x} \, dx, \tag{2}$$

has already been shown to converge uniformly in any interval bounded from above. If, then, $\alpha'$ be an arbitrary value of $\alpha$, it can be included within such an interval, and hence $\psi(\alpha)$ is continuous at $\alpha'$. But $\alpha'$ is any value of $\alpha$. Hence the integral is continuous for all values of $\alpha$.

Turning now to the first integral,

$$\varphi(\alpha) = \int_0^{1} x^{\alpha-1} e^{-x} \, dx, \tag{3}$$

we see that this is an improper integral which can be transformed into the class considered in §§ 9-13 by a change of variable,

$$t = \frac{1}{x}, \qquad x = \frac{1}{t}:$$

$$\varphi(\alpha) = \int_1^{\infty} t^{-\alpha-1} e^{-\frac{1}{t}} \, dt. \tag{4}$$

This integral converges uniformly in any interval $0 < \delta \leqq \alpha < \infty$. Hence it represents a function continuous for all positive values of $\alpha$.

Combining the two results we see that $\Gamma(\alpha)$ is continuous for all positive values of $\alpha$.

*Differentiation of the Gamma Function.* The function $\psi(\alpha)$ has a derivative given by Leibniz's Rule:

$$\frac{d\psi}{d\alpha} = \int_1^\infty x^{\alpha-1} e^{-x} \log x \, dx.$$

The integral 4) can be differentiated under the sign of integration for positive values of $\alpha$. Hence, on transforming back, we see that the same is true of the integral 3).

Thus it appears that the Gamma Function has a continuous derivative for all positive values of $\alpha$, given by the formula:

$$5) \qquad \frac{d\Gamma}{d\alpha} = \int_0^\infty x^{\alpha-1} e^{-x} \log x \, dx.$$

The existence of higher derivatives of all orders is proved in like manner:

$$6) \qquad \frac{d^n\Gamma}{d\alpha^n} = \int_0^\infty x^{\alpha-1} e^{-x} (\log x)^n \, dx.$$

**§ 16. Improper Integrals over a Finite Interval.** Instead of transforming the integral 3) into the form 4) considered in §§ 9-13 it is possible to give an independent treatment, parallel to that of the earlier case.

Let $f(x, \alpha)$ be continuous in the region

$$R: \qquad a < x \leqq b, \qquad A \leqq \alpha \leqq B,$$

though not in general bounded. The integral

$$\int_a^b f(x, \alpha) \, dx$$

is said to *converge uniformly* in the interval $A \leqq \alpha \leqq B$ if, to an arbitrary $\epsilon > 0$ there corresponds a positive $\delta$, independent of $\alpha$, such that

$$\left|\int_{x'}^{x''} f(x, \alpha)\, dx\right| < \epsilon, \qquad a < x', x'' < a + \delta.$$

The two theorems corresponding to Theorems I, II, § 9 are now formulated as before. The $\mu(x)$-test is developed, and the theorems of continuity (§ 11), reversal of the order of integration (§ 12), and Leibniz's Rule follow as before. It is a useful exercise for the student to write these theorems out in detail and to give a complete and independent proof of each.

Similar remarks apply to the integral

$$\int_a^b f(x, \alpha)\, dx,$$

where $f(x, \alpha)$ is continuous in the region

$R$: $\qquad a \leqq x < b, \qquad A \leqq \alpha \leqq B.$

If $f(x, \alpha)$ is continuous in the region

$R$: $\qquad a < x < b, \qquad A \leqq \alpha \leqq B,$

the integral may be broken up into the sum :

$$\int_a^c + \int_c^b, \qquad a < c < b,$$

and each of the latter integrals treated as above.

*Remark.* It would be a mistake to think that the above definitions exclude the case of proper integrals. The improper integrals are analogous to infinite series, the proper integrals to sums, so that a proper integral, under the above definitions, is like an infinite series whose terms, from a definite point on, are all 0. Because of this analogy the improper integrals are sometimes called "infinite integrals", but the irrelevant connotations of such a terminology are too disturbing.

## EXERCISES

1. Show that

$$\int_g^h dy \int_0^1 x^{y-1}\, dx = \int_0^1 dx \int_g^h x^{y-1}\, dy, \qquad 0 < g < h,$$

and hence

$$\int_0^1 \frac{x^{h-1} - x^{g-1}}{\log x}\, dx = \log \frac{h}{g}.$$

2. Prove the last equation to be true by means of partial differentiation and Leibniz's Rule.

3. Let $f(x, \alpha)$ be continuous in the interval

$$a < x < A$$

for each $\alpha$ of a point set $A$, and let $\alpha_0$ be a point of condensation of $A$. Let

$$\lim_{(x', \alpha)=(x, \alpha_0)} f(x, \alpha) = \omega(x)$$

and let

$$\int_a^A f(x, \alpha)\, dx$$

converge uniformly in $A$. Then

$$\lim_{\alpha=\alpha_0} \int_a^A f(x, \alpha)\, dx = \int_a^A \lim_{\alpha=\alpha_0} f(x, \alpha)\, dx.$$

**§ 17. The Beta-Function.** The Beta-Function is defined by the integral

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx. \tag{1}$$

It converges for $0 < m$, $0 < n$, and is a proper integral when $1 \leqq m$, $1 \leqq n$. Break the integral up into the sum:

$$\int_0^c + \int_c^1, \qquad 0 < c < 1.$$

The first integral,

$$\varphi(m, n) = \int_0^c x^{m-1}(1-x)^{n-1}\, dx, \tag{2}$$

converges uniformly in every region $0 < \delta \leqq m$, $0 < n$. For, if

$$\mu(x) = x^{\delta-1}(1-x)^{-1},$$

then

$$0 \leqq x^{m-1}(1-x)^{n-1} \leqq \mu(x),$$

and the $\mu(x)$-Test applies.

Similarly, the second integral,

$$3) \qquad \psi(m, n) = \int_{c}^{1} x^{m-1}(1-x)^{n-1}\,dx,$$

converges uniformly in every region $0 < m$, $0 < \delta \leqq n$. Hence the Beta Function is continuous throughout the region $0 < m$, $0 < n$.

By a change of variable,

$$y = \frac{x}{1-x}, \qquad x = \frac{y}{1+y},$$

the Beta Function can also be written in the form:

$$4) \qquad B(m, n) = \int_{0}^{\infty} \frac{y^{m-1}\,dy}{(1+y)^{m+n}}.$$

It is connected with the Gamma Function by the relation, § 19:

$$5) \qquad B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

### EXERCISE

Show that the Beta Function admits derivatives of all orders.

**§ 18. Both Limits Infinite.** There remains the case of the reversal of the order integration in the integral

$$1) \qquad \int_{a}^{\infty} dx \int_{b}^{\infty} f(x, y)\,dy.$$

Let $f(x, y)$ be continuous in the region

$$R: \qquad a \leqq x, \qquad b \leqq y.$$

We can obtain a test by means of the Theorem of Chap. V, § 10. Let

$$s(x, y) = \int_{a}^{x} dx \int_{b}^{y} f(x, y)\,dy.$$

**Theorem.** *Let*

i)
$$\int_b^\infty f(x,y)\,dy$$

*converge uniformly in any finite interval,* $a \leqq x \leqq A$*;*

ii)
$$\int_a^\infty f(x,y)\,dx$$

*converge uniformly in any finite interval* $b \leqq y \leqq B$*;*

iii)
$$\int_a^\infty dx \int_b^y f(x,y)\,dy$$

*converge uniformly in the infinite interval,* $b \leqq y < \infty$.

*Then each of the integrals*

$$\int_b^\infty dy \int_a^\infty f(x,y)\,dx, \qquad \int_a^\infty dx \int_b^\infty f(x,y)\,dy$$

*converges, and the two are equal:*

$$\int_b^\infty dy \int_a^\infty f(x,y)\,dx = \int_a^\infty dx \int_b^\infty f(x,y)\,dy.$$

*Proof.* From i) it follows that

$$\lim_{y=\infty} s(x,y) = \int_b^\infty dy \int_a^x f(x,y)\,dx = \int_a^x dx \int_b^\infty f(x,y)\,dy = \varphi(x)$$

exists.

From ii) it follows that

$$\lim_{x=\infty} s(x,y) = \int_a^\infty dx \int_b^y f(x,y)\,dy = \int_b^y dy \int_a^\infty f(x,y)\,dx = \psi(y)$$

exists.

From iii) it follows, since

$$s(x'',y) - s(x',y) = \int_{x'}^{x''} dx \int_b^y f(x,y)\,dy,$$

that

$$|s(x'', y) - s(x', y)| < \epsilon, \qquad g \leqq x', x'',$$

where $g$ is independent of $y$. Hence all the hypotheses of the Theorem cited are fulfilled and consequently:

a) $$\lim_{x=\infty} \varphi(x) \text{ exists, or}$$

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy \quad \text{converges;}$$

b) $$\lim_{y=\infty} \psi(y) \text{ exists, or}$$

$$\int_b^\infty dy \int_a^\infty f(x, y)\, dy \quad \text{converges;}$$

c) $$\int_a^\infty dx \int_b^\infty f(x, y)\, dy = \int_b^\infty dy \int_a^\infty f(x, y)\, dx.$$

Thus the theorem is proved.

COROLLARY[*]. *If, in particular,*

$$0 \leqq f(x, y),$$

*and if*

$$\int_a^\infty dx \int_b^\infty f(x, y)\, dy$$

*converges, Condition* iii) *is automatically fulfilled.*

For, let

$$\mu(x) = \int_b^\infty f(x, y)\, dy$$

Then $\mu(x)$ is continuous and $\geqq 0$. Moreover,

$$\int_a^\infty \mu(x)\, dx = \int_a^\infty dx \int_b^\infty f(x, y)\, dy$$

converges. Hence the de la Vallée-Poussin $\mu(x)$-test applies to the integral

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[*] This corollary, which is of chief importance in practice, is due to Professor C. A. Shook.

$$\int_a^\infty F(x,y)\,dx, \qquad F(x,y)=\int_b^y f(x,y)\,dy,$$

since

$$0 \leqq \int_b^y f(x,y)\,dy \leqq \int_b^\infty f(x,y)\,dy = \mu(x).$$

**§ 19. Application.** *The B-Function in Terms of the* Γ*-Function.* The $B$-function can be expressed in terms of the Γ-function by the formula:

$$B(m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}. \tag{1}$$

The formal part of the proof is easily given. If in the Γ-integral, § 15, we change the variable of integration from $x$ to $yx$ we have:

$$\Gamma(m)=\int_0^\infty y^m x^{m-1} e^{-yx}\,dx.$$

Thus

$$\Gamma(m)\,y^{n-1}e^{-y}=\int_0^\infty x^{m-1}y^{m+n-1}e^{-y(1+x)}\,dx,$$

and

$$\Gamma(m)\int_0^\infty y^{n-1}e^{-y}\,dy=\int_0^\infty x^{m-1}\,dx\int_0^\infty y^{m+n-1}e^{-y(1+x)}\,dy, \tag{2}$$

provided it is permissible to reverse the order of integration in the iterated integral. The value of the integral on the left is $\Gamma(n)$. The first integral to be computed on the right is substantially the Γ-integral. For if in § 15, 1), we change the variable of integration from $x$ to $(1+x)y$, we find:

$$\int_0^\infty y^{m+n-1}e^{-y(1+x)}\,dy=\frac{\Gamma(m+n)}{(1+x)^{m+n}}.$$

Thus

$$\Gamma(m)\,\Gamma(n)=\Gamma(m+n)\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}}.$$

But this last integral, by § 17, 4), is equal to $B(m, n)$.

It remains to justify the reversal of order in the iterated integral 2). Consider first the iterated integral:

$$3)\qquad \int_1^\infty dx \int_1^\infty x^{m-1} y^{m+n-1} e^{-y(1+x)}\, dy.$$

In accordance with Condition i) of the theorem of § 18 the integral

$$\int_1^\infty x^{m-1} y^{m+n-1} e^{-y(1+x)}\, dy$$

converges uniformly in any finite interval $1 \leqq x \leqq G$. For,

$$x^{m-1} y^{m+n-1} e^{-y(1+x)} < G^{m-1} y^{m+n-1} e^{-y} = \mu(y),$$

and the $\mu$-test applies.

Secondly, the integral

$$\int_1^\infty x^{m-1} y^{m+n-1} e^{-y(1+x)}\, dx$$

converges uniformly in any finite interval $1 \leqq y \leqq G$. For

$$x^{m-1} y^{m+n-1} e^{-y(1+x)} < G^{m+n-1} x^{m-1} e^{-x} = \mu(x),$$

and the $\mu$-test applies.

Since the integrand is always positive and one iterated integral converges, Condition iii) is automatically fulfilled; i. e. we have the case considered in the Corollary.

Thus the right to reverse the order of integration in the iterated integral 3) is established. Turning now to the integral 2), which is the one we are interested in:

$$4)\qquad \int_0^\infty dx \int_0^\infty x^{m-1} y^{m+n-1} e^{-y(1+x)}\, dy,$$

we see that we can break it up into the sum of four integrals:

$$\int_0^\infty\!\!\int_0^\infty = \int_0^1\!\!\int_0^1 + \int_1^\infty\!\!\int_0^1 + \int_0^1\!\!\int_1^\infty + \int_1^\infty\!\!\int_1^\infty$$

The last integral is the one we have just discussed, 3). Each of the others, by a suitable change of variable of integration, can be brought under the case considered in the Theorem of § 18. Thus in the first integral on the right we may replace $x$ by $1/x$ and $y$ by $1/y$. We find:

$$\int_1^\infty dx \int_1^\infty x^{-m-1}\, y^{-m-n-1}\, e^{-y(1+\frac{1}{x})}\, dy.$$

Here, the exponential factor does not help in the convergence — naturally. But since we are integrating from 1 to ∞, there is no difficulty in obtaining each time a suitable $\mu$-function. The further details are left to the reader.

The example suggests the formulation of a general theorem, to which we now turn.

**§ 20. Rectangular Region of Integration.** Consider the integral:

$$\int_a^A dx \int_b^B f(x, y)\, dy,$$

where $f(x, y)$ is continuous in the open rectangle

$R$: $$a < x < A, \qquad b < y < B,$$

but is not necessarily bounded. We can paraphrase the theorem of § 18 as follows.

THEOREM. *Let*

i) $$\int_b^B f(x, y)\, dy$$

*converge uniformly in any interval*

$$a' \leqq x \leqq A', \qquad -a < a' < A' < A;$$

ii) $$\int_a^A f(x, y)\, dx$$

*converge uniformly in any interval*

$$b' \leqq y \leqq B', \qquad -b < b' < B < B;$$

iii) $$\int_a^A dx \int_\eta^y f(x, y)\, dy$$

*converge uniformly in the region* $b < y < B,\ b < \eta < B.$

*Then each of the integrals*

$$\int\limits_q^B dy \int\limits_a^A f(x, y)\, dx, \qquad \int\limits_a^A dx \int\limits_b^B f(x, y)\, dy$$

*converges, and the two are equal:*

$$\int\limits_b^B dy \int\limits_a^A f(x,y)\, dx = \int\limits_a^A dx \int\limits_b^B f(x, y)\, dy.$$

The proof can be given by paraphrasing the proof of § 18, using the theorem of the Exercise, Chap. V, § 10,

$$s(\xi, x; \eta, y) = \int\limits_\eta^y dy \int\limits_\xi^x f(x, y)\, dx$$

where $\lim\,(\xi, x) = (a, A)$, $\lim\,(\eta, y) = (b, B)$.*

A less elegant, but more elementary, proof consists in breaking the given integral up into four integrals:

$$\int\limits_a^A \int\limits_b^B = \int\limits_\alpha^A \int\limits_\beta^B + \int\limits_a^\alpha \int\limits_\beta^B + \int\limits_\alpha^A \int\limits_b^\beta + \int\limits_a^\alpha \int\limits_b^\beta, \qquad \left\{ \begin{array}{l} a < \alpha < A \\ b < \beta < B \end{array} \right.$$

and then transforming each into the integral considered in § 18. Thus the first integral on the right will be subjected, for example, to the change of variable:

$$x' = \frac{x - \alpha}{A - x}, \qquad y' = \frac{y - \beta}{B - y}.$$

And similarly in the case of each of the other integrals.

COROLLARY, *If, in particular,* $f(x, y) \geqq 0$, *and if*

$$\int\limits_a^A dx \int\limits_b^B f(x, y)\, dy$$

*converges, Condition* iii) *of the hypothesis is automatically satisfied.*

Finally, one or both of the limits of integration $A$, $B$ may be replaced by $+\infty$, and independently either or both of the limits $a$, $b$ by $-\infty$.

---

* The proof was given in this way by one of my students at Harvard, whose name I cannot now recall.

**§ 21. Appraisal of an Alternating Integral.** Consider the integral

$$1) \qquad \int_c^\infty \varphi(x)\sin x\,dx,$$

where $\varphi(x)$ is continuous, $c \leqq x$, and

$$\varphi(x') \geqq \varphi(x''), \qquad c \leqq x' < x''.$$

$$\lim_{x=\infty} \varphi(x) = 0.$$

Write:

$$\int_c^x = \int_c^{x_1} + \int_{x_1}^{x_2} + \cdots + \int_{x_n}^{x},$$

where $x_k = k\pi$ and

$$x_1 - \pi \leqq c < x_1, \qquad x_n \leqq x < x_n + \pi.$$

Assume first that $c = x_1 - \pi = x_0$. We then have an alternating series whose terms are in general decreasing numerically (never increasing) and the general term approaches 0 as $x$ becomes infinite. Hence the integral converges. Moreover, the error made by breaking off with an arbitrary $x \geqq c$ does not exceed

$$\left| \int_{x_0}^{x_1} \varphi(x)\sin x\,dx \right| < 2\varphi(c),$$

or:

$$2) \qquad \left| \int_c^{x} \varphi(x)\sin x\,dx \right| < 2\varphi(c), \qquad c \leqq x < \infty.$$

The same appraisal holds when $x_1 - \pi < c < x_1$. For, extend the definition of $\varphi(x)$ to the interval $(x_0, c)$, setting

$$\varphi(x) = \varphi(c), \qquad x_1 - \pi \leqq x < c.$$

Then, for the extended function,

$$\int_c^{x} \varphi(x)\sin x\,dx$$

lies between

$$\int_{x_0}^{x} \varphi(x) \sin x \, dx \qquad \text{and} \qquad \int_{x_1}^{x} \varphi(x) \sin x \, dx,$$

and each of these integrals is appraised by 2).

With the same conditions for $\varphi(x)$ the same appraisal is obtained for the integral

$$\int_{c}^{\infty} \varphi(x) \sin(x + \gamma) \, dx.$$

In particular,

$$3) \qquad \left| \int_{c}^{x} \varphi(x) \cos x \, dx \right| < 2\varphi(c).$$

Finally, $m > 0$:

$$4) \qquad \left| \int_{c}^{x} \varphi(x) \sin mx \, dx \right| < \frac{2\varphi(c)}{m};$$

$$5) \qquad \left| \int_{c}^{x} \varphi(x) \cos mx \, dx \right| < \frac{2\varphi(c)}{m}.$$

### EXERCISE

Let $\varphi(x, \alpha)$ be defined in the region

$R$: $\qquad c \leqq x < \infty, \qquad 0 < \alpha \leqq A.$

Let $\varphi(x, \alpha_0)$ be a continuous function of $x$ for an arbitrary choice of $\alpha_0 > 0$. Let

i) $\quad \varphi(x', \alpha_0) \geqq \varphi(x'', \alpha_0), \qquad c \leqq x' < x'';$

ii) $\quad \lim\limits_{x=\infty} \varphi(x, \alpha_0) = 0;$

iii) $\quad \lim\limits_{\alpha=0} \varphi(c, \alpha) = 0.$

Then

$$\omega(\alpha) = \int_{c}^{\infty} \varphi(x, \alpha) \cos x \, dx$$

approaches 0, as $\alpha$ approaches 0.

## §22. Computation of $\int\limits_0^\infty \frac{\sin x}{x}\,dx.$

That this important integral converges, is shown in the Calculus. Let its value be denoted by $K$. Since

$$\tfrac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}},$$

we see that

$$\int\limits_0^\pi \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}\,dx = \frac{\pi}{2}.$$

Moreover, on changing the variable of integration from $x$ to $(n+\frac{1}{2})x$, we have:

$$K = \int\limits_0^\infty \frac{\sin x}{x}\,dx = \int\limits_0^\infty \frac{\sin(n+\frac{1}{2})x}{x}\,dx.$$

Hence

$$K - \frac{\pi}{2} = \int\limits_0^\pi \left(\frac{1}{x} - \frac{1}{2\sin\frac{1}{2}x}\right)\sin(n+\tfrac{1}{2})x\,dx + \int\limits_\pi^\infty \frac{\sin(n+\frac{1}{2})x}{x}\,dx.$$

Now, change the variable in the last integral, setting

$$t = (n+\tfrac{1}{2})x, \qquad h = (n+\tfrac{1}{2})\pi.$$

Thus this integral becomes:

$$\int\limits_h^\infty \frac{\sin t}{t}\,dt,$$

and so approaches 0 as $n = \infty$.

The first integral approaches 0 by Chap. VIII, § 1, Exercise 1 Hence $K = \pi/2$:

$$\int\limits_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

## §23. Applications.

The integral

1) $$u = \int\limits_0^\infty \frac{\cos mx}{1+x^2}\,dx$$

can be evaluated by setting

$$\frac{1}{1+x^2} = 2\int_0^\infty y\, e^{-y^2(1+x^2)}\, dy$$

and reversing the order of integration in the iterated integral

2) $$u = 2\int_0^\infty dx \int_0^\infty y\, e^{-y^2(1+x^2)} \cos mx\, dy.$$

For, by Example 1, § 14,

$$2\int_0^\infty y\, e^{-y^2x^2} \cos mx\, dx = \sqrt{\pi}\, e^{-\frac{m^2}{4y^2}}$$

and hence, by Example 2, § 14,

3) $$u = \sqrt{\pi}\int_0^\infty e^{-y^2-\frac{m^2}{4y^2}}\, dx = \frac{\pi}{2}\, e^{-|m|}.$$

It remains to justify the reversal of order in the iterated integral. This is done by the theorem of § 18.

ad i) $$\int_0^\infty y\, e^{-y^2(1+x^2)} \cos mx\, dy$$

converges uniformly, $0 \leqq x$, for

$$|\, y\, e^{-y^2(1+x^2)} \cos mx \,| \leqq y\, e^{-y^2} = \mu\,(y).$$

ad ii) $$\int_0^\infty y\, e^{-y^2(1+x^2)} \cos mx\, dx.$$

This is not so easy, for the uniform convergence cannot be established by means of the $\mu$-test.

For simplicity of presentation set $m = 1$ and consider the integral

4) $$\int_0^\infty y\, e^{-y^2x^2} \cos x\, dx.$$

This is an alternating integral, and the value of the remainder can thus be appraised. The integral has the value:

$$\int_0^{x_1} + \int_{x_1}^{x_2} + \cdots$$

where

$$x_k = k\pi - \frac{\pi}{2}.$$

Since the function

$$y\,e^{-y^2x^2}, \qquad 0 < y,$$

is monotonic decreasing as $x$ increases and has the limit 0, the value of the partial remainder

$$\int\limits_{\xi}^{x'} e^{-y^2x^2}\cos x\,dx, \qquad \frac{\pi}{2} \leqq \xi < x',$$

is less numerically than the contribution of the arch in which $\xi$ appears:

$$a \leqq \xi < b, \qquad a = k\pi - \tfrac{\pi}{2}, \qquad b = k\pi + \tfrac{\pi}{2},$$

or: $a = x_k$, $b = x_{k+1}$. Now,

$$\left|\int\limits_a^b e^{-y^2x^2}\cos x\,dx\right| < \int\limits_a^b e^{-y^2x^2}\,dx = \pi e^{-y^2X^2}, \qquad a < X < b.$$

Hence

$$5) \qquad \left|\int\limits_{\xi}^{x'} e^{-y^2x^2}\cos x\,dx\right| < \pi e^{-y^2x_k^2}.$$

For a fixed value of $k$ (however large) and a small value of $y > 0$ this appraisal will not be small; but it will always be less than $\pi$.

Let us formulate now what we wish to establish. To prove that the integral 4) converges uniformly in any finite interval $0 \leqq y \leqq B$ is to show that to an arbitrary $\epsilon > 0$ corresponds a $g$ independent of $y$ such that

$$6) \qquad \left|\int\limits_{\xi}^{x} y\,e^{-y^2(1+x^2)}\cos x\,dx\right| < \epsilon, \qquad g \leqq \xi, x.$$

We can do this as follows. First

$$\int\limits_{\xi}^{x} e^{-y^2x^2}\cos x\,dx$$

is bounded for all values of $0 \leqq \xi,\ x < \infty$:

$$\left| \int_{\xi}^{x} e^{-y^2x^2} \cos x \, dx \right| < M. \tag{7}$$

Consequently

$$\left| \int_{\xi}^{x} y \, e^{-y^2(1+x^2)} \cos x \, dx \right| < yM,$$

no matter how $\xi$, $x$ be chosen. Let $\delta$ be determined by the relation:

$$M\delta = \epsilon. \tag{8}$$

Then the condition 6) is satisfied when $0 \leqq y \leqq \delta$, no matter how $\xi$, $x$ be chosen.

Next, restrict $y$ to the interval

$$\delta < y \leqq B, \tag{9}$$

And now the appraisal 5) shows us that, if $g$ be so chosen that

$$\pi B \, e^{-g^2\delta^2} < \epsilon,$$

the condition 6) will be fulfilled. This completes the proof of uniform convergence under ii) when $m \neq 0$.

Turning now to iii) we have to show that

$$\int_{0}^{\infty} F(x, y) \, dx$$

converges uniformly in the interval $0 \leqq y < +\infty$, where

$$F(x, y) = \int_{0}^{y} y \, e^{-y^2(1+x^2)} \cos mx \, dy.$$

It is easy to find here a function $\mu(x)$, namely:

$$\mu(x) = \int_{0}^{\infty} y \, e^{-y^2(1+x^2)} \, dy = \frac{1}{2(1+x^2)} \cdot$$

This completes the proof when $m \neq 0$. When $m = 0$, the evaluation 3) is seen at once to hold by inspection of the integral 1).

## Exercises

1. Show that

$$\int_0^\infty \frac{x \sin m x}{1+x^2}\,dx = \begin{cases} \frac{\pi}{2} e^{-m}, & 0 < m \\ 0, & m = 0 \\ -\frac{\pi}{2} e^{m}, & m < 0 \end{cases}$$

2. Show that

$$\int_0^\infty \frac{\sin m x}{x}\,dx = \begin{cases} \frac{\pi}{2}, & 0 < m; \\ 0, & m = 0 \\ -\frac{\pi}{2}, & m < 0 \end{cases}$$

suggestion: Set $m = 1$,

$$\frac{1}{x} = \int_0^\infty e^{-yx}\,dy,$$

and apply the theorem of § 20.

This is, of course, an exercise in method. The result can be obtained at once from § 22.

3. Show that

$$\int_0^\infty \frac{y \sin k y}{b^2 + y^2}\,dy = \int_0^\infty \frac{k \cos b x}{k^2 + x^2}\,dx, \qquad k \neq 0,$$

by establishing the right to reverse the order in the iterated integral

$$\int_0^\infty dx \int_0^\infty e^{-yx} \cos b x \sin k y\,dy.$$

4. Show that

$$\int_0^\infty \frac{x \sin b x}{k^2 + x^2}\,dx = \begin{cases} \frac{\pi}{2} e^{-|bk|}, & 0 < b \\ 0, & b = 0 \\ -\frac{\pi}{2} e^{-|bk|}, & b < 0 \end{cases}$$

**§ 24. Duhamel's Theorem.** In formulating certain physical quantities as limits of sums it frequently happens that the sum in question is nearly of the form of a sum whose limit is a definite integral, and it seems highly probable from physical considerations that the two variables have the same limit; namely, the definite integral. Duhamel devised a theorem which meets the requirement.

THEOREM 1. *Let*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n \qquad 0 \leqq \alpha_k ,$$

*be a sum of infinitesimals which approaches a limit as* $n = \infty$. *Let*

$$\beta_1 + \beta_2 + \cdots + \beta_n$$

*be a second sum such that*

$$\lim \frac{\beta_k}{\alpha_k} = 1$$

*in the sense of a uniform approach; i. e.* $\beta_k/\alpha_k$ *shall approach the limit* 1 *uniformly as* $n = \infty$. *Then the second sum approaches a limit, and the two limits are equal:*

$$\lim_{n=\infty} (\alpha_1 + \alpha_2 + \cdots + \alpha_n) = \lim_{n=\infty} (\beta_1 + \beta_2 + \cdots + \beta_n).$$

*Proof*. Let

$$\frac{\beta_k}{\alpha_k} = 1 + \zeta_k.$$

Then, by hypothesis, to a positive $\epsilon$ chosen at pleasure there corresponds a *fixed* $m$ such that

$$|\zeta_k| < \epsilon, \qquad m \leqq n.$$

Hence

$$\beta_1 + \beta_2 + \cdots + \beta_k = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$
$$+ \alpha_1 \zeta_1 + \alpha_2 \zeta_2 + \cdots + \alpha_n \zeta_n.$$

Since $\alpha_k > 0$, we have:

$$|\alpha_1 \zeta_1 + \alpha_2 \zeta_2 + \cdots + \alpha_n \zeta_n| < (\alpha_1 + \alpha_2 + \cdots + \alpha_n)\ \epsilon.$$

The sum $\Sigma\, \alpha_k$ is bounded, and so this last expression can be made as small as we please. This completes the proof.

Another form of the theorem is the following. Consider a proper-definite integral:

$$\lim_{n=\infty} \sum_{k=1}^{n} f(x'_k)\, \Delta x_k = \int_a^b f(x)\, dx,$$

the integrand being continuous in the closed interval $a \leqq x \leqq b$.

Let $\varphi_k$ be defined for each partition of. the interval and for each value of $k$; and let

$$\varphi_k = f(x_k) + \zeta_k,$$

where $\zeta_k$ approaches 0 uniformly; i. e. to a positive $\epsilon$ chosen at pleasure there shall correspond a *fixed* $\delta$ such that

$$|\zeta_k| < \epsilon, \qquad |\Delta x_k| < \delta,$$

no matter how the interval $(a, b)$ may be partioned. Then the sum:

$$\varphi_1 \Delta x_1 + \varphi_2 \Delta x_2 + \cdots + \varphi_n \Delta x_n$$

approaches a limit, and this limit is equal to the definite integral:

$$\lim_{n=\infty} \sum_{k=1}^{n} \varphi_k \Delta x_k = \int_a^b f(x)\, dx.$$

From a theoretical standpoint this latter theorem is more general. But in practice the earlier one is adequate, and more convenient to apply. Cf. the Author's *Introduction to the Calculus*, pp. 301-307. We may formulate the result as follows.

THEOREM 2. *Let $\mathfrak{A}$ be a closed regular region of the n-dimensional space of the variables $(x_1, \cdots, x_n)$, and let $f(x_1, \cdots, x_n)$ be continuous in $\mathfrak{A}$. Consider the integral:*

$$\underbrace{\int \cdots \int}_{\mathfrak{A}} f\, dr = \lim_{n=\infty} \sum_{k=1}^{n} f_k \Delta r_k.$$

*Let $\varphi_k$ be defined for each element of volume, and let*

$$\varphi_k = f_k + \zeta_k,$$

*where $\zeta_k$ approaches 0 uniformly. Then*

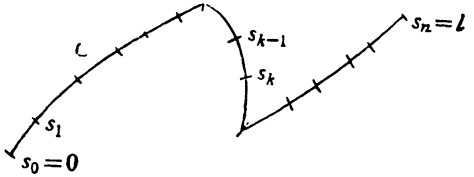
$$\sum_{k=1}^{n} \varphi_k \Delta r_k$$

*approaches a limit, and*

$$\lim_{n=\infty} \sum_{k=1}^{n} \varphi_k \Delta r_k = \underbrace{\int \cdots \int}_{\mathfrak{A}} f\, dr.$$

In attaching the name of *Duhamel* to these theorems one recalls the man who first dealt constructively with the question stated at the beginning of the paragraph. Duhamel himself formulated Theorem 1, without however recognizing the importance of the *uniform* restriction — this question had not been raised in his time. On the other hand, Theorem 2 is only one of a group of theorems designed by followers of Duhamel to accomplish the same object. It might be better to refer to this whole group of theorems as *Duhamel's Principle.*

**§ 25. Line Integrals.** Let $C$ be a regular curve in the $(x, y)$-plane, and let $f$ be a function, defined in the points of $C$ and continuous. Thus $f$ is a continuous function of the length of the arc of $C$, measured from an extremity. Divide $C$ up in any manner into $n$ arcs as indicated, and form the sum:



$$\sum_{k=1}^{n} f_k \Delta s_k,$$

where $\Delta s_k = s_k - s_{k-1}$ and $f_k$ is the value of $f$ at an arbitrary point $s_k'$ of the $k$-th arc. Then the *line integral* of the function $f$ along $C$ is defined and denoted as follows:

$$1) \qquad \lim_{n=\infty} \sum_{k=1}^{n} f_k \Delta s_k = \int_C f\, ds,$$

the longest arc $\Delta s_k$ approaching 0. It is nothing more or less, in substance, than the ordinary definite integral:

$$2) \qquad \int_0^l f(s)\, ds.$$

But in form the definition is important; for, first, we are thinking of a function as defined *along a curve*, and not in an interval of the scale of numbers. And secondly there is no question of the sense of integration along $C$. We could equally well have measured $s$ from the other end of the curve, or from a third point, in either direction. It is essential that $\Delta s_k$ be taken *absolutely*, not as a signed quantity. It is important to point out these facts at this stage, since the value of the line integral 6) below depends essentially on the sense in which it is extended along the curve.

THEOREM. *Let* $\varphi_k$ *be defined in any manner, corresponding to the k-th element of arc,* $\Delta s_k$, *and let*

$$\varphi_k = f(s_k) + \zeta_k,$$

*where* $\zeta_k$ *approaches* 0 *uniformly as* $n = \infty$. *Then the sum*

$$\sum_{k=1}^{n} \varphi_k \Delta s_k$$

*approaches a limit as* $n = \infty$, *the maximum* $\Delta s_k$ *approaching* 0, *and*

3) $$\lim_{n=\infty} \sum_{k=1}^{n} \varphi_k \Delta s_k = \int_C f\,ds.$$

The proof follows at once from Duhamel's Theorem, § 24.

An important application in practice is the following. Let $\varphi_k$ be defined as above, and let $l_k$ denote the length of the chord. Then the sum

$$\sum_{k=1}^{n} \varphi_k l_k$$

approaches a limit as $n = \infty$ and

4) $$\lim_{n=\infty} \sum_{k=1}^{n} \varphi_k l_k = \int_C f\,ds.$$

For, let an arc of $C$ be represented by the equations:

$$x = \psi(s), \quad y = \omega(s);$$

$$\psi'(s)^2 + \omega'(s)^2 = 1.$$

Then

$$l_k^2 = \Delta x_k^2 + \Delta y_k^2 =$$

$$[\,\psi'(s_k + \theta\Delta s_k)^2 + \omega'(s_k + \theta'\Delta s_k)^2\,]\,\Delta s_k^2.$$

Here,

5) $$\psi'(s_k + \theta\Delta s_k) = \psi'(s_k) + \eta_k, \qquad \omega'(s_k + \theta'\Delta s_k) = \omega'(s_k) + \eta_k'$$

where $\eta_k$, $\eta'_k$ are uniformly small:

$$|\eta_k| < \epsilon \quad \text{and} \quad |\eta'_k| < \epsilon \quad \text{if} \quad \Delta s_k < \delta;$$

$\delta$, independent of $s_k$. It follows, then, that

$$l_k^2 = [1 + 2\psi'(s_k)\eta_k + 2\omega'(s_k)\eta'_k + \eta_k^2 + \eta_k'^2]\Delta s_k^2.$$

The functions $\psi'(s)$, $\omega'(s)$ are bounded.

$$|\psi'(s)| < M, \qquad |\omega'(s)| < M.$$

And, of course,

$$l_k \leqq \Delta s_k.$$

Hence

$$(1 - 2M\epsilon)\,\Delta s_k^2 < l_k^2 \leqq \Delta s_k^2.$$

From this result it appears that

$$l_k = (1 + \zeta'_k)\,\Delta s_k, \qquad |\zeta'_k| < \frac{M\epsilon}{\sqrt{1 - 2M\epsilon}},$$

provided $\Delta s_k < \delta$. We infer, then, that

$$\varphi_k l_k = [f(s_k) + \zeta_k](1 + \zeta'_k)\,\Delta s_k$$

$$= [f(s_k) + \zeta''_k]\,\Delta s_k,$$

where $\zeta''_k$ can be made uniformly small by a suitable choice of $\delta$, and hence the convergence comes under the case treated in the Theorem.

## §26. Continuation. The Integral: $\int\limits_C P\,dx + Q\,dy$.

Let $P$ and $Q$ be defined and continuous along the curve $C$, and let $C$ be divided into $n$ arcs by the points $(x_k, y_k)$, $k = 0, 1, \cdots, n$. Let

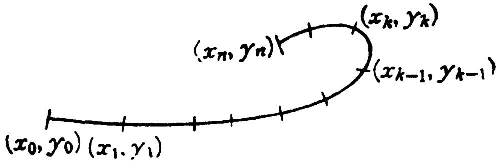
$$\Delta x_k = x_k - x_{k-1}, \qquad \Delta y_k = y_k - y_{k-1},$$

and form the sum:

$$\sum_{k=1}^{n} [P(x_k, y_k)\,\Delta x_k + Q(x_k, y_k)\,\Delta y_k].$$

Let $n = \infty$, the longest $\Delta s_k$ approaching the limit 0. Then the sum approaches a limit, and this limit is defined as the *line integral of $P\,dx + Q\,dy$, taken along $C$*:

$$6)\quad \lim_{n=\infty} \sum_{k=1}^{n} [P(x_k, y_k)\,\Delta x_k + Q(x_k, y_k)\,\Delta y_k] = \int_C P\,dx + Q\,dy.$$



The proof of convergence is as follows. Measure the arc of $C$ from the point $(x_0, y_0)$. Then

$$P(x_k, y_k)\,\Delta x_k + Q(x_k, y_k)\,\Delta y_k =$$
$$\left\{ P(x_k, y_k)\frac{\Delta x_k}{\Delta s_k} + Q(x_k, y_k)\frac{\Delta y_k}{\Delta s_k} \right\} \Delta s_k.$$

The brace differs uniformly little from

$$P(x_k, y_k)\cos\tau_k + Q(x_k, y_k)\sin\tau_k,$$

because of 5), and hence the convergence is ensured by the results of § 25. We see, moreover, that

$$7)\quad \int_C P\,dx + Q\,dy = \int_0^l (P\cos\tau + Q\sin\tau)\,ds,$$

where $s$ is measured from $(x_0, y_0)$ and $\tau$ is the angle from the positive axis of $x$ to the tangent in the sense of the increasing $s$.

Unlike the line integral 1) the present line integral depends on the *sense* in which the curve $C$ is described; for, the point $(x_0, y_0)$ may be taken at either extremity. Another notation for this integral is:

$$8)\quad \int_{(a,b)}^{(A,B)} P\,dx + Q\,dy,$$

where $(a, b) = (x_0, y_0)$ are the coordinates of the initial point, and $(A, B)$ are the coordinates of the terminal point. We see, then, that

$$9)\qquad \int_{(A,B)}^{(a,b)} P\,dx + Q\,dy = -\int_{(a,b)}^{(A,B)} P\,dx + Q\,dy.$$

The extension of the definition of each line integral, 1) and 6), to the case of $n$ dimensions is immediate:

$$\int_C P\,dx + Q\,dy + R\,dz,$$

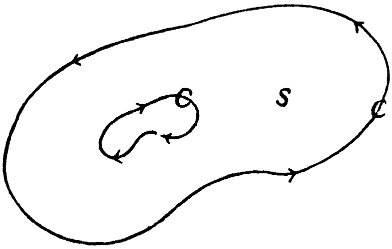
and

$$\int_C P_1\,dx_1 + \cdots + P_n\,dx_n.$$

*Green's Theorem.** Let $P = P(x, y)$ be continuous in a region $S$ and let $\partial P/\partial y$ exist and be continuous in the interior of $S$, and bounded. Then

$$\int_S\int \frac{\partial P}{\partial y}\,dS = -\int_C P\,dx,$$

the line integral being extended in the positive sense over the complete boundary $C$ of $S$; cf. *Advanced Calculus*, p. 222. Similarly,



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* The Germans call it "Gauss's Theorem" — and with equal justification. For Gauss, like Green, perceived its fundamental importance in analysis. But the one name cannot be preferred to the other on the basis of priority, since the representation (in three dimensions) of volume integrals by surface integrals goes back to Lagrange (1760/61): *Oeuvres* vol. 1, p. 263.

$$\int_S\int \frac{\partial Q}{\partial x}\,dS = \int_C Q\,dy.$$

From these two equations it follows that

$$\text{10)}\qquad \int_S\int \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dS = -\int_C P\,dx + Q\,dy.$$

The proof of these theorems given in the Calculus is complete, provided the boundary is cut by a parallel to an axis of coordinates in a bounded number of points and line segments. It is possible to remove this restriction from the identity 10), which is the only one that concerns us here, without going into an intricate discussion of a limiting process, provided all the first partial derivatives of $P$ and $Q$ exist and are continuous within $S$, and are bounded. For, the identity 10) is invariant of a rotation of the axes, or even, more generally, of an affine transformation. Let

$$\text{11)}\qquad \begin{cases} x' = a_1 x + b_1 y + c_1 \\ y' = a_2 x + b_2 y + c_2 \end{cases} \qquad a_1 b_2 - a_2 b_1 = 0;$$

and let $P'$, $Q'$ be determined by the transformation:

$$\text{12)}\qquad \begin{cases} P = a_1 P' + a_2 Q' \\ Q = b_1 P' + b_2 Q' \end{cases}$$

Then

$$P'\,dx' + Q'\,dy' = P\,dx + Q\,dy$$

and

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = (a_1 b_2 - a_2 b_1)\left(\frac{\partial P'}{\partial y'} - \frac{\partial Q'}{\partial x'}\right).$$

Moreover,

$$J = \frac{\partial(x', y')}{\partial(x, y)} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

If $J > 0$, then

$$\text{13)}\quad \int_{S'}\int \left(\frac{\partial P'}{\partial y'} - \frac{\partial Q'}{\partial x'}\right) dS' = \int_S\int \left(\frac{\partial P'}{\partial y'} - \frac{\partial Q'}{\partial x'}\right) \frac{\partial(x', y')}{\partial(x, y)}\,dS$$

$$= \int_S\int \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dS.$$

On the other hand,

$$14)\qquad \int_{C'} P'\,dx' + Q'\,dy' = \int_{C} P\,dx + Q\,dy,$$

each line integral being extended in the positive sense around the complete boundary of its respective region. Thus the invariance of the identity 10) is established in this case.

If, however, $J < 0$, Equation 13) is replaced by the equation:

$$15)\qquad \iint_{S'} \left(\frac{\partial P'}{\partial y'} - \frac{\partial Q'}{\partial x'}\right) dS' = -\iint_{S} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dS.$$

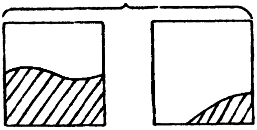
On the other hand, Equation 14) is replaced by the equation:

$$16)\qquad \int_{C'} P'\,dx' + Q'\,dy' = -\int_{C} P\,dx + Q\,dy,$$

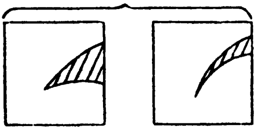
since the positive sense of integration over the boundary of $S'$ is the opposite of that in which the image of $C$ is described. Thus the theorem is true in this case, too.

*Partition of S into Regions of Normal Type.* It is possible to divide a region $S$ into a finite number of *regions of normal type.* These consist i) of squares with their sides parallel to the coordinate axes and not exceeding in length a given fixed quantity, $h$, and moreover, lying inside of $S$; ii) of regions defined by the following figures:

Type I



Type II



The bounding curve in Type I can be expressed in the form:

$$y = f(x);$$

and the two curves in Type II are grven by the equations:

$$y = f(x), \qquad y = \varphi(x),$$
$$\varphi(x) < f(x), \qquad a < x \leqq b,$$

where $f(x)$, $\varphi(x)$ are continuous, together with their first derivatives, in an interval

$$a \leqq x \leqq b, \qquad b - a < h.$$

Each of these figures can be rotated through any multiple of 90°.*

It may happen that the boundary $C$ of $S$ is cut by a parallel to an axis of coordinates in an infinite number of points and line segments. But on rotating a given region of normal type through a convenient angle, the new figure will have a boundary which is cut by such parallels at most in a bounded number of points.

For the transformed figure the proof of the identity 10) is that given in the calculus. By the theorem of invariance just established the identity holds for the original figure. And now the identity 10) for the entire region $S$, with its boundary $C$, is obtained by writing it down for each region of normal type, and summing.

*The Condition*: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. Let $S$ be a closed regular region. Let $P$ and $Q$ be continuous in $S$, and let their first partial derivatives exist and be continuous at all interior points of $S$, and bounded in $S$.

Let $\Sigma$ be any regular closed region contained in $S$. Consider the integral:

$$\int\limits_{\Gamma} P\,dx + Q\,dy, \tag{17}$$

extended in the positive sense over the complete boundary $\Gamma$ of $\Sigma$. From the identity 10) it follows that, if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{18}$$

at every interior point of $S$, then

$$\int\limits_{\Gamma} P\,dx + Q\,dy = 0. \tag{19}$$

Conversely, if Equation 19) holds for an arbitrary $\Sigma$ then 18) is true throughout the interior of $S$.

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* A detailed proof of this theorem of partition is given in the Author's *Funktionentheorie*, vol. I, Chap. V, § 9.

These results serve as the basis of the discussion of the function $F(x, y)$ defined by the line integral:

$$F(x, y) = \int_{(a,b)}^{(x,y)} P\,dx + Q\,dy, \tag{20)}$$

$P$ and $Q$ satisfying the condition 18); cf. the *Advanced Calculus*, pp. 222-233, and the *Funktionentheorie*, vol. I, Chap. IV.

# Chapter X

## The Gamma Function

**§ 1. Definition.** The Gamma Function has been defined in Chap. IX, § 15 by means of the integral:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{1}$$

The integral converges absolutely for all positive values of $x$ and uniformly in every interval $0 < \delta \leqq x \leqq G$, where $G$ is arbitrarily large and $\delta$ is arbitrarily small. Thus $\Gamma(x)$ is continuous for all positive values of $x$.

The function possesses a derivative, given by Leibniz's Rule:

$$\frac{d\Gamma}{dx} = \int_0^\infty t^{x-1} e^{-t} \log t\, dt. \tag{2}$$

Moreover,

$$\frac{d^2\Gamma}{dx^2} = \int_0^\infty t^{x-1} e^{-t} (\log t)^2\, dt, \tag{3}$$

and so we see that

$$0 < \frac{d^2\Gamma}{dx^2} \tag{4}$$

for all positive values of $x$. Hence the graph of the function:

$$y = \Gamma(x), \tag{5}$$

is concave upward for all positive values of $x$. Furthermore,

$$\Gamma(0^+) = +\infty, \qquad \Gamma(+\infty) = +\infty. \tag{6}$$

The first relation follows from the fact that, for small values of $x$.

$$\Gamma(x) > \int_0^1 t^{x-1} e^{-1}\, dt = \frac{1}{ex}.$$

The second relation is a result of the fact (cf. § 2) that

7) $$\Gamma(n+1) = n!,$$

combined with the fact just established, that the curve is concave upward.

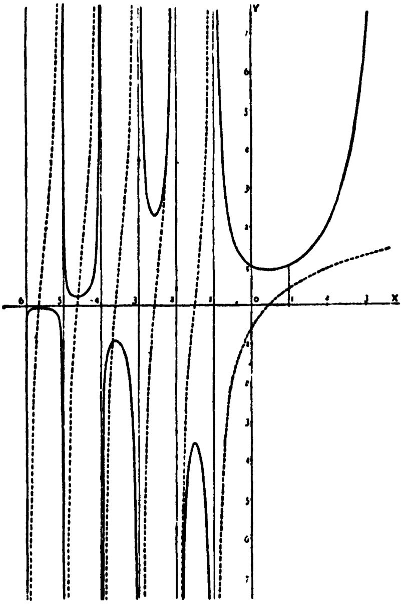
Finally, we note that, from the definition 1):

8) $$\Gamma(x) > 0, \quad 0 < x.$$

Also

9) $$\Gamma(1) = 1, \quad \Gamma(2) = 1.$$

The first of these last relations is proved by direct evaluation of the integral 1). The second follows from 7) by putting $n = 1$.



The figure shows the graph of the function

$$y = \Gamma(x+1),$$

the dotted curve representing the function $d \log \Gamma(x+1)/dx$; cf. Duval, *Annals of Math.* 2d. ser. (1903/04) vol. 5.

The graph of 5) is thus seen to have one and only one minimum, and this occurs for a value of $x$ between 1 and 2. The value of $x$ has been found to be: $x = 1.46163 \cdots$

**§ 2. The Difference Equation.** The Gamma Function obeys the law:

$$1)\qquad \Gamma(x+1) = x\,\Gamma(x).$$

This is known as the *difference equation.* It is proved at once by integration-by parts:

$$\int t^{x-1} e^{-t}\,dt = \frac{t^x e^{-t}}{x} + \frac{1}{x}\int t^x e^{-t}\,dt, \qquad 0 < x.$$

A first application of the difference equation consists in setting $x$ equal successively to the natural numbers: $x = 1, 2, 3, \cdots$ and observing that $\Gamma(1) = 1$, cf. § 1, 9).

A second application enables us to define $\Gamma(x)$ for negative values of $x$. Write

$$2)\qquad \Gamma(x) = \frac{\Gamma(x+1)}{x}.$$

The right-hand side of this equation has a meaning when $-1 < x < 0$. This shall be the definition of $\Gamma(x)$ for $-1 < x < 0$. It thus appears that

$$\Gamma(x) < 0, \qquad -1 < x < 0,$$

and continuous, the graph having the lines $x = 0$ and $x = -1$ as asymptotes.

Repeating the process, setting $-2 < x < -1$ in 2), we define $\Gamma(x)$ in the latter interval. The graph of the function is shown in the figure. In any panel, $-(k+1) < x < -k$, the curve is always concave downward when $k$ is even, and concave upward when $k$ is odd. The proof is given conveniently by means of a later result; cf. § 3.

**§ 3. Gauss's Product.** Gauss[*] based his treatment of the Gamma Function on the following product, which he denoted by $\Pi(n, x)$. We shall show in § 4 that the limiting function is the function $\Gamma(x)$ defined in § 1. For the present we shall write with Gauss:

$$1)\qquad \Pi(n, x) = \frac{1\cdot 2\cdot \cdots \cdot (n-1)}{x(x+1)(x+2)\cdots(x+n-1)}\,n^x$$

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[*] *Werke*, vol. III, p. 144. The date is January 30, 1812.

where $x$ has any real* value $\neq 0, -1, -2, \cdots$, and prove that, as $n$ becomes infinite, $\Pi(n, x)$ approaches a limit. This limit we will denote by $\Gamma(x)$:

$$\Gamma(x) = \lim_{n=\infty} \Pi(n, x). \tag{2}$$

*The Convergence Proof.* The variable $\Pi(n, x)$ can be written in the form:

$$\Pi(n, x) = \Pi(2, x)\frac{\Pi(3, x)}{\Pi(2, x)}\frac{\Pi(4, x)}{\Pi(3, x)}\cdots\frac{\Pi(n, x)}{\Pi(n-1, x)} \tag{3}$$

$$= \frac{1}{x}\cdot\frac{2^x}{x+1}\cdot\frac{2\cdot 3^x}{(x+2)2^x}\cdot\frac{3\cdot 4^x}{(x+3)3^x}\cdots\frac{(n-1)n^x}{(x+n-1)(n-1)^x}.$$

If, then, we set

$$f_n(x) = \frac{n(n+1)^x}{(x+n)n^x} = \frac{(1+\frac{1}{n})^x}{1+\frac{x}{n}}, \tag{4}$$

we have

$$\lim_{n=\infty} \Pi(n, x) = \frac{1}{x}\prod_{n=1}^{\infty} f_n(x), \tag{5}$$

and the question of the convergence of $\Pi(n, x)$ becomes the question of the convergence of this infinite product. But we have methods for dealing with this latter question; cf. Chap. I, § 10.

Consider the series of logarithms:

$$\sum_{n=m}^{\infty} \log f_n(x).$$

Here,

$$\log f_n(x) = x \log\left(1 + \frac{1}{n}\right) - \log\left(1 + \frac{x}{n}\right).$$

The convergence of this series is established at once by comparing its terms with the corresponding terms of the known convergent series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

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* The convergence proof applies at once to the complex domain, $x$ being replaced by $z = x + y\sqrt{-1}$.

and applying the theorem of Chap. 1, § 2:

$$\lim_{n=\infty} \frac{\log f_n(x)}{1/n^2} = \frac{x^2 - x}{2}.$$

The value of the limit is not important — only the fact that a limit exists.

Thus the convergence of $\Pi(n, x)$ is established and a function $\Gamma(x)$ is defined by 2), or:

$$\text{6)} \qquad \Gamma(x) = \frac{1}{x} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^x}{1 + \frac{x}{n}}$$

*Properties of* $\Gamma(x)$. Since

$$\Pi(n, 1) = 1$$

for all values of $n$ it follows that

$$\text{7)} \qquad \Gamma(1) = 1.$$

Next, observe that

$$\Pi(n, x + 1) = \frac{nx}{x + n} \Pi(n, x).$$

Allowing $n$ to become infinite we have the *Difference Equation*:

$$\text{8)} \qquad \Gamma(x + 1) = x\,\Gamma(x).$$

Notice that this result holds for *all* values of $x \neq 0, -1, -2, \cdots$. In particular,

$$\text{9)} \qquad \Gamma(n + 1) = n!, \qquad n = 1, 2, 3, \cdots.$$

A further relation satisfied by the $\Gamma$-function is obtained from the product:

$$\Pi(n, x)\,\Pi(n, -x) = -\frac{\{(n-1)!\}^2}{x^2(1 - x^2)(2^2 - x^2)\cdots([n-1]^2 - x^2)}$$

$$= -\frac{1}{x}\,\frac{1}{x\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\cdots\left(1 - \frac{x^2}{(n-1)^2}\right)}$$

Allow $n$ to increase. The denominator corresponds to the infinite product for the sine-function, Chap. VI, § 6. Thus

$$\text{10)} \qquad \Gamma(x)\,\Gamma(-x) = -\frac{\pi}{x \sin \pi x}.$$

Another form of this relation is:

11) $$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

In particular, from this last relation follows that

12) $$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

*Differentiation.* The Γ-function is given by the series:

13) $$\log \Gamma(x) = -\log x + \sum_{n=1}^{\infty}\left\{x \log\left(1+\frac{1}{n}\right) - \log(x+n) + \log n\right\},$$

when $x > 0$. The term-by-term derivative series represents in all cases the logarithmic derivative:

14) $$\frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} + \sum_{n=1}^{\infty}\left\{\log\left(1+\frac{1}{n}\right) - \frac{1}{x+n}\right\}.$$

This latter series converges uniformly in any interval

$$-G \leqq x \leqq G,$$

from which the points $x = 0, -1, -2, \cdots$ have been removed. For

$$\log\left(1+\frac{1}{n}\right) - \frac{1}{x+n} = \left\{\log\left(1+\frac{1}{n}\right) - \frac{1}{n}\right\} + \frac{x}{n(n+x)}.$$

If, then, we choose the $M_n$ of Weierstrass's $M$-test as follows:

$$M_n = \left|\log\left(1+\frac{1}{n}\right) - \frac{1}{n}\right| + \frac{G}{n(n-G)}, \qquad G < n,$$

we see that this series converges by comparing $M_n$ with $1/n^2$.

It is now easy to complete the proof that $\Gamma(x)$ has a derivative for all values of $x$ for which the function is defined. Moreover, the Γ-function has a second derivative given by differentiating 14):

15) $$\frac{\Gamma''(x)}{\Gamma(x)} = \left(\frac{\Gamma'(x)}{\Gamma(x)}\right)^2 + \sum_{n=0}^{\infty}\frac{1}{(x+n)^2}.$$

Since the right-hand side is always positive, it follows that $\Gamma''(x)$ and $\Gamma(x)$ always have the same sign. Thus the graph, § 2, is concave upward (downward) when $\Gamma(x)$ is positive (negative).

*Euler's Constant.* The development 6) can be replaced by the following:

16)
$$\Gamma(x) = \frac{e^{-Cx}}{x} \prod_{n=1}^{\infty} \frac{1}{\left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}},$$

where $C$ is Euler's Constant:

17)
$$C = \lim_{n=\infty} \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right\}.$$

For, the product in 6) has the value:

$$\prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^x e^{-\frac{x}{n}}}{\left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}}.$$

This product can be written as the quotient of two infinite products. Since

$$\log \left\{ \left(1 + \frac{1}{n}\right)^x e^{-\frac{x}{n}} \right\} = -x \left[ \frac{1}{n} - \log\left(1 + \frac{1}{n}\right) \right],$$

it is clear that the numerator product has the value $e^{-Cx}$. The value of $C$ is:

18)
$$C = 0.57721\ 56649\ 01532\ 86060 \cdots$$

Because of 16) we can write:

19)
$$\frac{1}{\Gamma(x)} = e^{Cx} x \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}$$

The product on the right converges for all values of $x$, real or complex, and it defines a so-called *entire* function of the complex variable, i.e. a function that is analytic for all finite values of the argument. In particular, it appears from either 16) or 19) that the $\Gamma$-function has no roots, even in the complex plane.

We have in the above another example of a *convergence factor*, whereby a divergent product,

$$\prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)$$

is rendered covergent:

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{x}{n}}$$

without disturbing the roots of the individual factors; cf. Chap. VI, § 6.

### Exercise

Prove the uniform convergence of the series 14) by means of the series 15).

**§ 4. Agreement of the Two Definitions.** From the relation:

$$\lim_{m=\infty}\left(1-\frac{t}{m}\right)^m = e^{-t}$$

it is easy to surmise that possibly

$$20)\qquad \lim_{m=\infty}\int_0^m t^{x-1}\left(1-\frac{t}{m}\right)^m dt = \int_0^\infty t^{x-1}e^{-t}\,dt = \Gamma(x):$$

The proof of the correctness of the surmise is not difficult. Let

$$f(t,m) = t^{x-1}\left(1-\frac{t}{m}\right)^m, \qquad 0 < t \leqq m;$$

$$f(t,m) = 0, \qquad m < t < \infty.$$

Then the first integral in 20) can be written:

$$21)\qquad \int_0^\infty f(t,m)\,dt.$$

Here:

$$\lim_{m=\infty} f(t,m) = t^{x-1}e^{-t}.$$

If, then, the integral 21) converges uniformly, the relation 20) results; cf. Chap. IX, § 20, Since

$$\left(1-\frac{t}{m}\right)^m = e^{m\log\left(1-\frac{t}{m}\right)} = e^{-t-\frac{t^2}{2m}-\frac{t^3}{3m^2}-\cdots}, \qquad 0 < t < m.$$

it is seen that

$$0 \leqq f(t, m) < t^{x-1} e^{-t}$$

for all values of $t$ and $m$. Thus the integral 21) satisfies a $\mu$-test:

$$\mu(t) = t^{x-1} e^{-t},$$

and the proof is complete.

The integral — we now change the notation from $m$ to $n$ —

$$\int_0^n t^{x-1}\left(1 - \frac{t}{n}\right)^n dt$$

can be evaluated explicitly and turns out to be equal to

$$\frac{1 \cdot \cdots \cdot n}{x(x+1)\cdots(x+n)} n^x.$$

For, change the variable of integration, setting $t = n\lambda$. Then

$$\int_0^n t^{x-1}\left(1 - \frac{t}{n}\right)^n dt = n^x \int_0^1 \lambda^{x-1}(1 - \lambda)^n d\lambda.$$

To this last integral apply the method of integration by parts:

$$\int_0^1 \lambda^{x-1}(1 - \lambda)^n d\lambda = \frac{n}{x} \cdot \frac{n-1}{x+1} \cdots \frac{1}{x+n-1} \int_0^1 \lambda^{x+n-1} d\lambda.$$

Thus finally

$$\int_0^n t^{x-1}\left(1 - \frac{t}{n}\right)^n dt = \frac{n}{x+n} \Pi(x, n).$$

Allow $n$ to become infinite. It appears, then, that

$$\int_0^\infty t^{x-1} e^{-t} dt = \lim \Pi(x, n).$$

Hence the two definitions lead to the same function, $\Gamma(x)$.

## § 5. Stirling's Formula.

For large values of $x$ the function $\Gamma(x)$ is extremely large. Its value can be computed approximately by the evaluation

1) $$\Gamma(x) = \sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x} e^{\varpi(x)},$$

where

$$2) \qquad 0 < \varpi(x) < \frac{1}{12x}$$

or

$$3) \qquad \varpi(x) = \frac{\theta}{12x}, \qquad 0 < \theta < 1.$$

Equation 1) is known as *Stirling's Formula.* We mention also the following evaluation for the factorial:

$$4) \qquad n! = \sqrt{2\pi}\; n^{n+\frac{1}{2}}\, e^{-n+\frac{\theta}{12n}}, \qquad 0 < \theta < 1.$$

An elementary proof of the truth of 1) where $0 \leqq \theta \leqq 1$ can be given as follows.

Let $\varpi(x)$ be defined by 1):

$$5) \qquad \varpi(x) = \log\Gamma(x) + x - (x - \tfrac{1}{2}) \log x - \log\sqrt{2\pi}.$$

Since

$$\Gamma(x+1) = x\,\Gamma(x)$$

it follows that

$$6) \qquad \varpi(x) - \quad (x+1) = \frac{2x+1}{2} \log\left(1 + \frac{1}{x}\right) - 1.$$

Remembering that

$$\log\frac{1+y}{1-y} = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots\right)$$

and setting

$$1 + \frac{1}{x} = \frac{1+y}{1-y}, \qquad y \doteq \frac{1}{2x+1},$$

we find:

$$\log\left(1 + \frac{1}{x}\right) = \frac{2}{2x+1}\left(1 + \frac{1}{3}\,\frac{1}{(2x+1)^2} + \frac{1}{5}\,\frac{1}{(2x+1)^4} + \cdots\right)$$

---

[*] cf. the excellent treatment of this subject by Godefroy, *Théorie élémentaire des séries*, Chap. VI, which we here reproduce. The reasoning does not show that $0 < \theta < 1$, but only that $0 \leqq \theta \leqq 1$. There is, of course, no loss in this less general theorem for purposes of computation, since $\theta$, in the more general form, might conceiveably lie so near to 0 or to 1 that the difference would have no effect on any approximate computation.

Hence

$$\frac{2}{2x+1} < \log\left(1+\frac{1}{x}\right) < \frac{2}{2x+1}\left(1+\frac{1}{12}\frac{1}{x(x+1)}\right).$$

It follows, then from 6) that

7) $$0 < \varpi(x) - \varpi(x+1) < \frac{1}{12}\left(\frac{1}{x} - \frac{1}{x+1}\right).$$

By a repeated application of this formula:

$$0 < \varpi(x+1) - \varpi(x+2) < \frac{1}{12}\left(\frac{1}{x+1} - \frac{1}{x+2}\right)$$

. . . . . . . . . . . . . . . . . . . . . . . . . . .

$$0 < \varpi(x+n-1) - \varpi(x+n) < \frac{1}{12}\left(\frac{1}{x+n-1} - \frac{1}{x+n}\right)$$

we see that

8) $$0 < \varpi(x) - \varpi(x+n) < \frac{1}{12}\left(\frac{1}{x} - \frac{1}{x+n}\right).$$

We now proceed to show that

9) $$\lim_{x=\infty} \varpi(x) = 0.$$

It will then follow from 8) on letting $n = \infty$ that[*]

10) $$0 \leqq \varpi(x) \leqq \frac{1}{12x}$$

or

10′) $$\varpi(x) = \frac{\theta}{12x}, \qquad 0 \leqq \theta \leqq 1.$$

*Proof of* 9). First, let $x = n$ in 8):

11) $$0 < \varpi(n) - \varpi(2n) < \frac{1}{24n}.$$

We now *introduce the function* $\psi(x)$:

12) $$\psi(x) = \frac{\Gamma(x)}{\sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x}}.$$

---

[*] Godefroy infers at this point that

$$0 < \varpi(x) < \frac{1}{12x}.$$

This relation is true, but his reasoning establishes only the less general relation 10) or 10′).

Then

13) $$\varpi(x) = \log \psi(x), \qquad \psi(x) = e^{\varpi(x)}.$$

Since

$$\frac{\psi(n)}{\psi(2n)} = e^{\varpi(n)-\varpi(2n)},$$

it follows from 11) that

$$1 < \frac{\psi(n)}{\psi(2n)} < e^{\frac{1}{24n}}.$$

Hence

14) $$\lim_{n=\infty} \frac{\psi(n)}{\psi(2n)} = 1.$$

Next, form the function

$$\frac{\psi(n)^2}{\psi(2n)}.$$

We shall show by direct computation that

15) $$\lim_{n=\infty} \frac{\psi(n)^2}{\psi(2n)} = 1.$$

It will follow, then, from 14) and 15) that

16) $$\lim_{n=\infty} \psi(n) = 1.$$

The proof is as follows. From the definition of $\psi(x)$ by 12) and the property of the $\Gamma$-function:

$$\Gamma(n+1) = n!$$

we have:

17) $$\frac{\psi(n)^2}{\psi(2n)} = \frac{1}{\sqrt{\pi n}} \frac{2 \cdot 4 \cdot 6 \cdots \quad 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}.$$

Recall Wallis's formula for $\pi$:

$$\frac{\pi}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\cdots$$

$$= \lim_{n=\infty} \left(\frac{2 \cdot 4 \cdot 6 \cdots \quad 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\right)^2 \frac{1}{2n+1}.$$

It thus appears that the right-hand side of 17) approaches 1, and thus 15) is established. We see, then, that 16) is true.

Finally, we will show that, for an arbitrary $x > 0$,

$$\lim_{n=\infty} \frac{\psi(n)}{\psi(x+n)} = 1. \tag{18}$$

We have:

$$\frac{\psi(n)}{\psi(x+n)} = \frac{(n-1)!\,e^n}{n^{n-\frac{1}{2}}} \cdot \frac{(x+n)^{x+n-\frac{1}{2}}}{\Gamma(x+n)\,e^{x+n}}.$$

Now

$$\frac{(x+n)^{x+n-\frac{1}{2}}}{n^{n-\frac{1}{2}}} = \left(1+\frac{x}{n}\right)^{x+n-\frac{1}{2}} n^x$$

and

$$\Gamma(x+n) = x(x+1)\cdots(x+n-1)\,\Gamma(x).$$

Remembering Gauss's product:

$$\Gamma(x) = \lim_{n=\infty} \frac{1\cdot 2\cdot\;\cdots\;\cdot(n-1)}{x(x+1)\cdots(x+n-1)}\, n^x$$

we infer at once the truth of 18).

From 16) and 18) it follows that, for an arbitrary value of $x > 0$,

$$\lim_{n=\infty} \psi(x+n) = 1.$$

Hence, from 13),

$$\lim_{n=\infty} \varpi(x+n) = 0,$$

and now 8) yields 10), hence 9) — the relation we set out to establish. The proof of 1) under the appraisal

$$0 \leqq \varpi(x) \leqq 1$$

or

$$\varpi(x) = \frac{\theta}{12x}, \qquad 0 \leqq \theta \leqq 1$$

is complete.

*Gudermann's Formula.* On writing down Equation 6) for $x$, $x+1, \cdots, x+n$ and adding we find:

$$\varpi(x) - \varpi(x+n+1) = \sum_{p=0}^{n} \left[\left(x + \frac{2p+1}{2}\right)\log\left(1+\frac{1}{x+p}\right) - 1\right]$$

Allowing $n$ to become infinite and remembering 9) we obtain Gudermann's Formula:

$$19)\qquad \varpi(x) = \sum_{n=0}^{\infty}\left[\left(x + \frac{2n+1}{2}\right)\log\left(1 + \frac{1}{x+n}\right) - 1\right].$$

***Binet's Series.*** The general term $u_n(x)$ in 19) can be written in the form :

$$u_n(x) = (x+n)\log\left(1 + \frac{1}{x+n}\right) - 1 + \frac{1}{2}\log\left(1 + \frac{1}{x+n}\right)$$

$$= -\frac{1}{2}\frac{1}{x+n} + \frac{1}{3}\frac{1}{(x+n)^2} - \frac{1}{4}\frac{1}{(x+n)^3} + \cdots$$

$$+ \frac{1}{2}\frac{1}{x+n} - \frac{1}{4}\frac{1}{(x+n)^2} + \frac{1}{6}\frac{1}{(x+n)^3} - \cdots$$

$$= \sum_{p=2}^{\infty}(-1)^p \frac{p-1}{(p+1)\cdot 2p}\frac{1}{(x+n)^p}$$

or

$$u_n(x) = \frac{1}{3\cdot 4}\frac{1}{(x+n)^2} - \frac{2}{4\cdot 6}\frac{1}{(x+n)^3} + \frac{3}{5\cdot 8}\frac{1}{(x+n)^4} - \cdots$$

By virtue of the theorem of Chap. VII, § 5 we have :

$$20)\qquad (x) = \frac{1}{3\cdot 4}\sum_{n=0}^{\infty}\frac{1}{(x+n)^2} - \frac{2}{4\cdot 6}\sum_{n=0}^{\infty}\frac{1}{(x+n)^3}$$

$$+ \frac{3}{5\cdot 8}\sum_{n=0}^{\infty}\frac{1}{(x+n)^4} - \cdots$$

Another form for $u_n(x)$ is the following :

$$u_n(x) = -(x+n+1)\log\left(1 - \frac{1}{x+n+1}\right)$$

$$+ \frac{1}{2}\log\left(1 - \frac{1}{x+n+1}\right) - 1.$$

This form leads to the development :

$$21)\qquad \varpi(x) = \frac{1}{3\cdot 4}\sum_{n=1}^{\infty}\frac{1}{(x+n)^2} + \frac{2}{4\cdot 6}\sum_{n=1}^{\infty}\frac{1}{(x+n)^3}$$

$$+ \frac{3}{5\cdot 8}\sum_{n=1}^{\infty}\frac{1}{(x+n)^4} + \cdots$$

## EXERCISES

1. Give all the details in the above deduction of Formulas 20) and 21).

2. By means of 20) and the evaluations of the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots, \qquad p = 2, 4, 6, \cdots;$$

$$\frac{1}{m^p} + \frac{1}{(m+1)^p} + \cdots$$

given in Chap. 1, § 4 and Chap. VII, § 7 obtain appraisals for $\varpi(m)$

# Chapter XI

## Fourier's Integral

**§1. Fourier's Integral. Heuristic Treatment.** Let $f(x)$ be continuous in the interval $-\infty < x < +\infty$ save for isolated values, and let

$$1) \qquad \int_{-\infty}^{\infty} |f(x)|\, dx$$

converge. Consider an arbitrary interval

$$2) \qquad -l < x < l.$$

By means of the transformation

$$3) \qquad \frac{y}{\pi} = \frac{x}{l}$$

the function goes over into a function of $y$:

$$4) \qquad f(x) = F(y),$$

having the same properties. In particular, the integrals that define the Fourier's coefficients of $F(y)$ will converge:

$$5) \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\cos ny\, dy, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\sin ny\, dy.$$

Thus a Fourier's development* of the function $F(y)$ exists:

$$6) \qquad \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos ny + b_n \sin ny)$$

or

$$7) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} F(s)\, ds + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} F(s)\cos n(s-y)\, ds.$$

We will now transform back to the variable $x$ by 3). Thus we are led to the development:

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* **We apply the term *development* to denote the series 6), irrespective of whether the series converges and represents the function.**

$$8)\qquad \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\cos\frac{n\pi}{l}(t-x)\,dt.$$

In order to bring out the central thought in the investigation that follows, let us confine ourselves for the present to those functions of the class before us, of which we know that they can be expanded into a Fourier's series; i.e. the series 7) and hence the series 8) shall converge at all points of continuity to the value of the function. Thus if $f(x)$ satisfies the conditions of §1, Chap. VIII, this will be the case.

*Formal Deduction of Fourier's Integral.* Let

$$9)\qquad \Delta\alpha = \frac{\pi}{l}.$$

The series in 8) now takes the form:

$$10)\qquad \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\cos n\,\Delta\alpha\,(t-x)\,\Delta\alpha\,dt.$$

If we consider a function $\varphi(\alpha)$ continuous for $0 \leqq \alpha$, and divide the interval $0 \leqq \alpha \leqq A$ into $m$ equal parts, $\Delta\alpha = A/m$, then the sum

$$11)\qquad \sum_{n=1}^{m}\varphi(\alpha_n)\,\Delta\alpha$$

approaches the limit

$$12)\qquad \int_{0}^{A}\varphi(\alpha)\,d\alpha$$

when $m = \infty$. The series

$$\sum_{n=1}^{\infty}\varphi(\alpha_n)\,\Delta\alpha$$

when $\Delta\alpha$ approaches 0, suggests the integral

$$\int_{0}^{\infty}\varphi(\alpha)\,d\alpha.$$

Thus the expression 10), when $l$ becomes infinite, suggests the integral:

$$13)\qquad \frac{1}{\pi}\int_{0}^{\infty}d\alpha\int_{-\infty}^{\infty}f(t)\cos\alpha\,(t-x)\,dt.$$

This is known as *Fourier's Integral.* It converges, under suitable restrictions, to the value $f(x)$.

**§2. A Lemma.** Lemma. *Let the function $\varphi(x)$ be continuous for $0 < x < +\infty$, save for isolated discontinuities, and let either*

$$\text{i)} \qquad \int_0^\infty \varphi(x)\,dx$$

*converge absolutely; or*

$$\text{ii}_1) \qquad \int_0^G \varphi(x)\,dx$$

*converge absolutely, where $G$ is an arbitrary positive constant;*

$\text{ii}_2)$ *$\varphi(x)$ decrease monotonically toward 0 as $x$ increases:*

$$\begin{cases} \varphi(x') \geqq \varphi(x''), & 0 \leqq A < x' < x''; \\ \lim\limits_{x=\infty} \varphi(x) = 0, \end{cases}$$

*where $A$ is some fixed number.*

*Let $\varphi(x)$ approach a limit when $x$ approaches 0 from above·*

$$\lim_{x=0^+} \varphi(x) = \varphi(0^+),$$

*and let*

$$\frac{\varphi(x) - \varphi(0^+)}{x}$$

*be bounded at the origin.*

*Then*

$$\int_0^\infty d\alpha \int_0^\infty \varphi(x) \cos \alpha x \, dx$$

*converges to the value $\frac{\pi}{2}\varphi(0^+)$.*

We will prove the lemma first for the simplest and most important case, namely, that $\varphi(x)$ has no discontinuities* and satisfies Condition i). Here the integrand (if for the present purpose $\varphi(x)$ is defined as $\varphi(0^+)$ for $x = 0$) is continuous in the region

$$R \qquad 0 \leqq x, \qquad -\infty < \alpha < +\infty,$$

and the integral

---

* The proof applies, however, to the case that $\varphi(x)$ has isolated discontinuities and is bounded at each one of them.

$$14)\qquad \int_0^\infty \varphi(x)\cos\alpha x\,dx$$

converges uniformly for all values of $\alpha$, as is seen by setting

$$\mu(x) = |\varphi(x)|.$$

Hence 14) represents a continuous function, and furthermore

$$15)\qquad \int_0^q d\alpha \int_0^\infty \varphi(x)\cos\alpha x\,dx = \int_0^\infty \varphi(x)\frac{\sin qx}{x}\,dx.$$

We proceed to show that the integral on the right of 15) converges toward $\frac{\pi}{2}\varphi(0^+)$ when $q$ tends to infinity. Write

$$16)\qquad \int_0^\infty \varphi(x)\frac{\sin qx}{x}\,dx = \varphi(0^+)\int_0^\infty \frac{\sin qx}{x}\,dx$$
$$+\int_0^\infty \frac{\varphi(x)-\varphi(0^+)}{x}\sin qx\,dx.$$

Since, by Chap. IX, § 22,

$$17)\qquad \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2},$$

we see on making the change of variable $t = qx$ that the first integral on the right of 16) has the value $\frac{\pi}{2}\varphi(0^+)$, no matter what value $q$ may have.

The second integral can be written:

$$18)\qquad \int_0^h \frac{\varphi(x)-\varphi(0^+)}{x}\sin qx\,dx +$$
$$\int_h^\infty \frac{\varphi(x)}{x}\sin qx\,dx - \varphi(0^+)\int_h^\infty \frac{\sin qx}{x}\,dx.$$

Let $\epsilon > 0$ be chosen arbitrarily. Then $h$ can be so determined that

$$\left|\int_h^\infty \frac{\varphi(x)}{x}\sin qx\,dx\right| \leqq \frac{1}{h}\int_0^\infty |\varphi(x)|\,dx < \epsilon.$$

Moreover,

$$\int_h^\infty \frac{\sin qx}{x}\,dx = \int_{qh}^\infty \frac{\sin t}{t}\,dt,$$

and so

$$\left| -\varphi(0^+) \int_h^\infty \frac{\sin qx}{x}\,dx \right| < \epsilon, \qquad \mu \leqq q.$$

As regards the first integral in 18), its limit is 0 as $q$ increases; cf. Chap, VIII, § 2, Ex. 3. Hence it remains numerically less that $\epsilon$ when $q \geqq \mu'$, and so the whole sum 18) is numerically less than $3\,\epsilon$. This completes the proof.

### Exercise

Let $\varphi(x, \xi)$ be continuous in the region

$$0 < x < \infty, \qquad A \leqq \xi \leqq B.$$

Let the limit

$$\lim_{x=0+} \varphi(x, \xi) = \varphi(0^+, \xi)$$

exist for each $\xi$. Let $\varphi(0^+, \xi)$ be bounded. Let the function

$$\Psi(x, \xi) = \frac{\varphi(x, \xi) - \varphi(0^+, \xi)}{x}$$

be bounded in the region

$$0 < x < c, \qquad A \leqq \xi \leqq B,$$

where $c$ is a positive constant. Finally, let the integral

$$\int_0^\infty |\varphi(x, \xi)|\,dx$$

converge and be bounded. Prove that the function of $(q, \xi)$ represented by the integral 15) converges uniformly in the interval $A \leqq \xi \leqq B$ when $q = \infty$.

Suggestion. Appraise the integral

$$\int_0^h \Psi(x, \xi) \sin qx\,dx$$

by assigning an arbitrary $\epsilon > 0$ and dividing the interval $0 \leqq x \leqq h$ by the points

$$x_0 = 0 < x_1 < \cdots < x_{n-1} < x_n = h,$$

so chosen that, if

$$\Psi(x, \xi) = \Psi(x_k, \xi) + \zeta_k, \quad x_k \leqq x \leqq x_{k+1}, \quad 0 \leqq k < n,$$

we have:

$$|\zeta_k| < \epsilon.$$

**§ 3. Continuation. The General Case*.** We turn now to the most general class of functions $\varphi(x)$ admitted by the Lemma, and proceed to establish 15), § 2:

$$19) \qquad \int_0^q d\alpha \int_0^\infty \varphi(x) \cos \alpha x \, dx = \int_0^\infty \varphi(x) \frac{\sin qx}{x} \, dx.$$

First and foremost it is seen that the integral

$$\int_0^h \varphi(x) \cos \alpha x \, dx, \qquad 0 < h,$$

converges, since $\varphi(x)$ is continuous except for isolated singularities and

$$\int_0^h |\varphi(x)| \, dx$$

converges. Consequently

$$\int_0^\infty \varphi(x) \cos \alpha x \, dx$$

converges in Case i) because it converges absolutely, and in Case ii) because it is an alternating integral.

The function defined by this integral,

$$20) \qquad \omega(\alpha) = \int_0^\infty \varphi(x) \cos \alpha x \, dx, \qquad 0 < \alpha,$$

is continuous. For

$$21) \quad \omega(\alpha_0 + \Delta\alpha) - \omega(\alpha_0) = \int_0^h \varphi(x) [\cos(\alpha_0 + \Delta\alpha) x - \cos \alpha_0 x] \, dx$$

---

* It is well to defer the study of this section till the rest of the chapter has been completed.

$$+\int\limits_h^\infty \varphi(x)\cos(\alpha_0+\Delta\alpha)\,x\,dx-\int\limits_h^\infty \varphi(x)\cos\alpha_0\,x\,dx,$$

$$0<\alpha_0, \qquad 0<\alpha_0+\Delta\alpha.$$

Each of these last two integrals can be made less numerically than $\epsilon$ by a suitable restriction of $\Delta\alpha$ and a proper choice of $h$, as follows.

In Case i) it is sufficient to choose $h$ so that

$$\int\limits_h^\infty |\varphi(x)|\,dx<\epsilon,$$

no restriction on $\Delta\alpha$ being needed.

In Case ii) let $\gamma$ be chosen so that $0<\gamma<\alpha_0$, and let $\Delta\alpha$ be subject to the restriction $\alpha_0+\Delta\alpha>\gamma$. Let $h \geqq A$, Chap. IX, § 21. Then, since,

$$22) \qquad \left|\int\limits_h^\infty \varphi(x)\cos\alpha x\,dx\right|<\frac{2\varphi(h)}{\alpha}, \qquad 0<\alpha,$$

it is enough to take $h$ so that

$$23) \qquad \frac{2\varphi(h)}{\gamma}<\epsilon.$$

Since $h$ is now a constant, it is possible further to restrict $\Delta\alpha$ so that the first integral in 21) remains numerically less than $\epsilon$, when $|\Delta\alpha|<\delta$; as is seen on removing from the interval of integration $(0, h)$ short segments including the points of discontinuity and applying to the contributions to the integral arising from these the appraisal:

$$24) \qquad \cos(\alpha_0+\Delta\alpha)\,x-\cos\alpha_0=-\Delta\alpha\sin(\alpha_0+\theta\,\Delta\alpha),$$

The proof of the continuity of $\omega(\alpha)$, $0<\alpha$, is herewith complete. $\omega(\alpha)$ is not, however, necessarily bounded, as is shown by the example

$$\int\limits_0^\infty \frac{\cos\alpha x}{\sqrt{x}}\,dx=\frac{1}{\sqrt{\alpha}}\int\limits_0^\infty \frac{\cos t}{\sqrt{t}}\,dt.$$

Next, we need to show that the order of integration can be reversed in the integral

$$25)\qquad \int_\gamma^q d\alpha \int_0^h \varphi(x) \cos \alpha x \, dx.$$

The proof is not difficult and can be left to the reader.

We are now prepared to show that the order of integration can be reversed in the integral

$$26)\qquad \int_\gamma^q d\alpha \int_0^\infty \varphi(x) \cos \alpha x \, dx, \qquad 0 < \gamma.$$

Write the inner integral in the form:

$$\int_0^\infty \varphi(x) \cos \alpha x \, dx = \int_0^h \varphi(x) \cos \alpha x \, dx + \int_h^\infty \varphi(x) \cos \alpha x \, dx.$$

Choose $g$ in Case i) so that

$$\int_g^\infty |\, \varphi(x) \,| \, dx < \epsilon;$$

in Case ii) so that $g \geqq A$ and

$$\frac{2\varphi(g)}{\gamma} < \epsilon.$$

Then

$$\left| \int_h^\infty \varphi(x) \cos \alpha x \, dx \right| < \epsilon, \qquad g \leqq h, \qquad \gamma \leqq \alpha.$$

Thus we see that

$$\left| \int_\gamma^q d\alpha \int_0^\infty \varphi(x) \cos \alpha x \, dx - \int_0^h \varphi(x) \, dx \int_\gamma^q \cos \alpha x \, d\alpha \right| =$$

$$\left| \int_\gamma^q d\alpha \int_h^\infty \varphi(x) \cos \alpha x \, dx \right| < (q - \gamma)\, \epsilon, \qquad g \leqq h.$$

This proves that the order of integration in the integral 26) can be reversed, and hence we infer the relation:

$$28)\qquad \int_\gamma^q d\alpha \int_0^\infty \varphi(x) \cos \alpha x \, dx =$$

$$\int_0^\infty \varphi(x)\frac{\sin qx}{x}\,dx-\int_0^\infty \varphi(x)\frac{\sin\gamma x}{x}\,dx.$$

The last integral in 28) approaches 0 with $\gamma$. For,

$$29)\quad \int_0^\infty \varphi(x)\frac{\sin\gamma x}{x}\,dx=\int_0^h \varphi(x)\frac{\sin\gamma x}{x}\,dx+\int_h^\infty \varphi(x)\frac{\sin\gamma x}{x}\,dx.$$

Let $\epsilon > 0$ be arbitrary. In Case i) it is sufficient to take $h$ so that

$$\int_h^\infty |\,\varphi(x)\,|\,dx < \epsilon.$$

In Case ii) $h \geqq A$ can be so chosen that

$$\left|\int_h^\infty \varphi(x)\frac{\sin\gamma x}{x}\,dx\right|=\left|\int_\tau^\infty \varphi(x)\frac{\sin t}{t}\,dt\right|<\varphi(h)\int_0^\pi\frac{\sin t}{t}\,dt<\epsilon.$$

And now, holding $h$ fast, we see that $\gamma$ can be so restricted that the first integral on the right of 29) remains numerically less than $\epsilon$.

If, then, in 28) we allow $\gamma$ to approach 0, the right-hand side approaches the first term as its limit. Hence the left-hand side converges and we have 19), or:

$$30)\qquad \int_0^q d\alpha\int_0^\infty \varphi(x)\cos\alpha x\,dx=\int_0^\infty \varphi(x)\frac{\sin qx}{x}\,dx.$$

The final step consists in showing that the integral on the right converges toward $\frac{\pi}{2}\,\varphi(0^+)$ as $q$ increases. The proof begins as in § 2 and we reach 18). In Case i) the last two integrals can be appraised as in the earlier proof. In Case ii) we have, by Chap. IX, § 21:

$$\left|\int_h^\infty \frac{\varphi(x)}{x}\sin qx\,dx\right| < \frac{2\varphi(h)}{qh},\qquad A \leqq h,$$

since the function $\varphi(x)/x$ decreases monotonically toward 0 as $x$ increases. It is sufficient, then, to choose $h$ so that

$$\frac{2\varphi(h)}{h} < \epsilon,\qquad A \leqq h.$$

The third integral in 18) is appraised as before, and so, in each case, there remains only the first integral, with $h$ fixed. With the aid of the Exercise of Chap. VIII, § 2, there is no difficulty in proving that this integral remains numerically less than $\epsilon$ when $q$ exceeds a suitable integer, $\mu \leqq q$, and the proof of the Lemma is now complete.

**§ 4. Convergence of Fourier's Integral. Differentiation.** The most important functions which can be represented by Fourier's integral belong to one or both of the following classes.

*Class* I. $f(x)$ is continuous except for isolated singularities and

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

converges.

*Class* II. $f(x)$ is continuous except for isolated singularities and

$\alpha$)
$$\int_{a}^{b} |f(x)|\, dx$$

converges, where $a$, $b$ are two arbitrary numbers;

$\beta$) $f(x)$ converges monotonically toward 0 when $x = +\infty$, and also (though not necessarily with the same sign) when $x = -\infty$.

THEOREM OF CONVERGENCE. *Let $f(x)$ be a function belonging either to Class* I. *or to Class* II. *Let $x$ be a point such that*

$$\lim_{t=x+} f(t) = f(x^+), \qquad \lim_{t=x-} f(t) = f(x^-).$$

*Let each of the difference-quotients:*

$$\frac{f(x+h) - f(x^+)}{h}, \qquad 0 < h < \delta;$$

$$\frac{f(x+h) - f(x^-)}{h}, \qquad -\delta < h < 0$$

*be bounded. Then*

$$\frac{1}{\pi}\int_{0}^{\infty} d\alpha \int_{-\infty}^{\infty} f(t) \cos \alpha (t-x)\, dt = \tfrac{1}{2}\left[f(x^+) + f(x^-)\right].$$

*If, in particular, the function is continuous at the point $x$, then*

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} d\alpha \int_{-\infty}^{\infty} f(t) \cos \alpha (t-x)\, dt.$$

The proof follows immediately from the Lemma. Make a change of variable:

$$u = t - x.$$

Then

$$\int_x^\infty f(t) \cos \alpha (t - x)\, dt = \int_0^\infty f(x + u) \cos \alpha u\, du.$$

Set

$$\varphi(u) = f(x + u),$$

and apply the Lemma.

Next, let

$$t - x = -u.$$

Then

$$\int_{-\infty}^x f(t) \cos \alpha (t - x)\, dt = \int_0^\infty f(x - u) \cos \alpha u\, du.$$

Set

$$\varphi(u) = f(x - u)$$

and apply the Lemma. Thus the proof results.

*Differentiation.* Let $f(x)$ be a function which meets all the conditions of the above theorem and, furthermore, is continuous without exception. Let it have a derivative which also satisfies the conditions of the theorem. Then, in an interval $a < x < b$ in which the derivative is continuous, it will be given by the integral:

$$f'(x) = \frac{1}{\pi} \int_0^\infty d\alpha \int_{-\infty}^\infty f'(t) \cos \alpha (t - x)\, dt.$$

For, we can write Fourier's Integral in the form:

$$f(x) = \frac{1}{\pi} \int_0^\infty d\alpha \int_0^\infty f(u + x) \cos \alpha u\, du +$$

$$\frac{1}{\pi} \int_0^\infty d\alpha \int_{-\infty}^0 f(u + x) \cos \alpha u\, du,$$

and each of these latter integrals can be differentiated by Leibniz's rule.

**§5. Derived Integrals.** It is possible to break the Fourier's integral up into two integrals as follows:

$$f(x) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty f(t)\cos\alpha(t-x)\,dt =$$

$$\frac{1}{\pi}\int_0^\infty \cos\alpha x\,d\alpha \int_{-\infty}^\infty f(t)\cos\alpha t\,dt + \frac{1}{\pi}\int_0^\infty \sin\alpha x\,d\alpha\int_{-\infty}^\infty f(t)\sin\alpha t\,dt.$$

In case $f(x)$ is an even function or an odd function these integrals can be simplified.

*Case* I. $f(x) = f(-x)$. Here the last integral vanishes and the first can be written in simpler form:

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\alpha x\,d\alpha \int_0^\infty f(t)\cos\alpha t\,dt.$$

*Case* II. $f(x) = -f(-x)$. Here

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\alpha x\,d\alpha \int_0^\infty f(t)\sin\alpha t\,dt.$$

*Example* 1. Let

$$f(x) = e^{-kx}, \qquad 0 \leqq x, \qquad 0 < k;$$

$$f(-x) = f(x).$$

This example comes under Case I. Since

$$\int_0^\infty e^{-kt}\cos\alpha t\,dt = \frac{k}{k^2+\alpha^2},$$

we have:

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{k\cos\alpha x}{k^2+\alpha^2}\,d\alpha, \qquad 0 \leqq x, \qquad 0 < k.$$

Thus

$$\int_0^\infty \frac{\cos\alpha x}{k^2+\alpha^2}\,d\alpha = \frac{\pi}{2|k|}e^{-|kx|}, \qquad k \neq 0.$$

*Example* 2. Let

$$f(x) = e^{-kx}, \qquad 0 \leqq x, \qquad 0 < k;$$

$$f(-x) = -f(x).$$

This example comes under Case II. Since

$$\int_0^\infty e^{-kt} \sin \alpha t \, dt = \frac{\alpha}{k^2 + \alpha^2},$$

we have:

$$e^{-kx} = \frac{2}{\pi} \int_0^\infty \frac{\alpha \sin \alpha x}{k^2 + \alpha^2} d\alpha, \qquad 0 < x, \qquad 0 < k.$$

Thus if $k \neq 0$:

$$\int_0^\infty \frac{\alpha \sin \alpha x}{k^2 + \alpha^2} d\alpha = \begin{cases} \frac{\pi}{2} e^{-|kx|}, & 0 < x; \\ 0, & x = 0; \\ -\frac{\pi}{2} e^{-|kx|}, & x < 0. \end{cases}$$

*Example* 3. Let

$$f(x) = x^{-k}, \qquad 0 < x, \qquad 0 < k < 1;$$

$$f(-x) = f(x).$$

Then

$$x^{-k} = \frac{2}{\pi} \int_0^\infty \cos \alpha x \, d\alpha \int_0^\infty t^{-k} \cos \alpha t \, dt.$$

Change the variable: $u = \alpha t$. Then

$$x^{-k} = \frac{2}{\pi} \int_0^\infty \alpha^{k-1} \cos \alpha x \, d\alpha \int_0^\infty u^{-k} \cos u \, du.$$

In particular, if $x = 1$:

$$\left[\int_0^\infty \frac{\cos \alpha \, d\alpha}{\alpha^{1-k}}\right] \left[\int_0^\infty \frac{\cos u \, du}{u^k}\right] = \frac{\pi}{2}.$$

Let $k = \frac{1}{2}$:

$$\int_0^\infty \frac{\cos x \, dx}{\sqrt{x}} = \pm \sqrt{\frac{\pi}{2}}.$$

Riemann has given the following ingenious determination of the ±-sign. Write

$$\int_0^\infty = \sum_{\mu=0}^{\infty} \int_{\mu\pi}^{(\mu+1)\pi}$$

and in the integral

$$\int_{\mu\pi}^{(\mu+1)\pi} \frac{\cos x}{\sqrt{x}}\, dx$$

make a change of variable :

$$x = \alpha + \mu\pi.$$

If, now, we set

$$F(\alpha) = \frac{1}{\sqrt{\alpha}} - \frac{1}{\sqrt{\alpha + \pi}} + \frac{1}{\sqrt{\alpha + 2\pi}} - \cdots,$$

we shall have :

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}\, dx = \int_0^\pi F(\alpha) \cos \alpha \, d\alpha,$$

provided the series can be integrated term by term. Now

$$\int_0^\pi = \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}$$

In the last integral change the variable of integration to $\pi - \alpha$. Thus we find :

$$\int_0^\pi F(\alpha) \cos \alpha \, d\alpha = \int_0^{\frac{\pi}{2}} [F(\alpha) - F(\pi - \alpha)] \cos \alpha \, d\alpha.$$

Now,

$$F'(\alpha) = -\frac{1}{2}\left[\frac{1}{(\sqrt{\alpha})^3} - \frac{1}{(\sqrt{\alpha + \pi})^3} + \cdots\right]$$

and so $F'(\alpha) < 0$. Hence

$$\frac{d}{d\alpha}[F(\alpha) - F(\pi - \alpha)] = F'(\alpha) + F'(\pi - \alpha) < 0,$$

and, since $F(\alpha) - F(\pi - \alpha) = 0$ when $\alpha = \pi/2$, we have:

$$0 < F(\alpha) - F(\pi - \alpha), \qquad 0 < \alpha < \frac{\pi}{2}.$$

Consequently the upper sign is to be chosen.

It is left to the student as an exercise to justify the integration and the differentiation of the series term by term.

*Example* 4. Let

$$f(x) = x^{-k}, \qquad 0 < x, \qquad 0 < k < 1;$$
$$f(-x) = -f(x).$$

Then

$$x^{-k} = \frac{2}{\pi}\int\limits_0^\infty \sin\alpha x\, d\alpha \int\limits_0^\infty t^{-k}\sin\alpha t\, dt.$$

Set $u = \alpha t$. Then

$$x^{-k} = \frac{2}{\pi}\int\limits_0^\infty \alpha^{k-1}\sin\alpha x\, d\alpha \int\limits_0^\infty u^{-k}\sin u\, du.$$

If $x = 1$,

$$\int\limits_0^\infty \frac{\sin\alpha\, d\alpha}{\alpha^{1-k}} \int\limits_0^\infty \frac{\sin u\, du}{u^k} = \frac{\pi}{2}.$$

Let $k = \frac{1}{2}$:

$$\int\limits_0^\infty \frac{\sin\alpha\, d\alpha}{\sqrt{\alpha}} = \sqrt{\frac{\pi}{2}}.$$

**§6. Fourier's Integral for Functions of Several Variables.** Let $f(x, y)$ be continuous, together with its derivatives of the first order, throughout the whole plane, and let the integrals

$$1) \qquad \int\limits_{-\infty}^\infty |f(x, y)|\, dy, \qquad \int\limits_{-\infty}^\infty |f(x, y)|\, dx$$

converge. Then from § 4 we have:

$$f(x, y) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty f(\xi, y) \cos\alpha(\xi - x)\, d\xi,$$

$$f(\xi, y) = \frac{1}{\pi}\int_0^\infty d\beta \int_{-\infty}^\infty f(\xi, \eta) \cos\beta(\eta - y)\, d\eta.$$

Hence

2)
$$f(x, y) =$$
$$\frac{1}{\pi^2}\int_0^\infty d\alpha \int_{-\infty}^\infty d\xi \int_0^\infty d\beta \int_{-\infty}^\infty f(\xi, \eta) \cos\alpha(\xi - x) \cos\beta(\eta - y)\, d\eta,$$

with a similar formula, in which the integrations with respect to $\alpha, \xi$ and $\beta, \eta$ are interchanged. In either of these forms, certain discontinuities of the function $f(x, y)$ and its derivatives along regular curves can be admitted. But the more important form of the Fourier's Integral is the following:

3)
$$f(x, y) =$$
$$\frac{1}{\pi^2}\int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^\infty d\xi \int_{-\infty}^\infty f(\xi, \eta) \cos\alpha(\xi - x) \cos\beta(\eta - y)\, d\eta.$$

This form can be obtained in the above restricted case of exceptionless continuity, provided the further requirements are laid down: —

The integrals

4)
$$\int_{-\infty}^\infty |f(x, y)|\, dy, \qquad \int_{-\infty}^\infty |f_x(x, y)|\, dy$$

shall converge uniformly in any finite interval, $a' \leqq x \leqq a''$. And similarly,

5)
$$\int_{-\infty}^\infty |f(x, y)|\, dx, \qquad \int_{-\infty}^\infty |f_y(x, y)|\, dx$$

shall converge uniformly, $b' \leqq y \leqq b''$. Finally, one of the integrals:

6) $$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x, y)|\, dy, \qquad \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |f(x, y)|\, dx$$

shall converge; then the other does, also.

The proof is as follows. It is sufficient to consider the integral

7) $$\int_0^\infty d\xi \int_0^\infty d\beta \int_0^\infty f(\xi, \eta) \cos \alpha\xi \cos \beta\eta\, d\eta.$$

Let

8) $$F(\xi, \beta) = \int_0^\infty f(\xi, \eta) \cos \alpha\xi \cos \beta\eta\, d\eta.$$

Then we wish to show that

9) $$\int_0^\infty d\xi \int_0^\infty F(\xi, \beta)\, d\beta = \int_0^\infty d\beta \int_0^\infty F(\xi, \beta)\, d\xi.$$

First of all, observe that

10) $$\int_0^\xi d\xi \int_0^\infty F(\xi, \beta)\, d\beta = \int_0^\infty d\beta \int_0^\xi F(\xi, \beta)\, d\xi.$$

For, the integral:

$$\int_0^\infty F(\xi, \beta)\, d\beta = \int_0^\infty d\beta \int_0^\infty \psi(\eta, \xi) \cos \beta\eta\, d\eta,$$

where

11) $$\psi(\eta, \xi) = f(\xi, \eta) \cos \alpha\xi,$$

converges uniformly in any bounded region, $0 \leqq \xi \leqq G$; cf. § 2, Exercise, p. 331. And $F(\xi, \beta)$ is continuous, $0 \leqq \xi$, $0 \leqq \beta$, because of 4).

Write

12) $$\Psi(\xi, \beta) = \int_0^\xi F(\xi, \beta)\, d\xi,$$

The integral:

$$\int_0^\infty F(\xi, \beta)\, d\xi$$

converges, since

$$|F(\xi,\beta)| \leqq \int_0^\infty |f(\xi,\eta)|\,d\eta.$$

This last integral defines a continuous function of $\xi$ by 4), and

$$\int_0^\infty d\xi \int_0^\infty |f(\xi,\eta)|\,d\eta$$

converges by 6). We can now write:

13) $$\Psi(\infty,\beta) = \int_0^\infty F(\xi,\beta)\,d\xi.$$

From 10) we have:

$$\int_0^\xi d\xi \int_0^\infty F(\xi,\beta)\,d\beta = \int_0^\infty \Psi(\xi,\beta)\,d\beta,$$

and from 13):

$$\int_0^\infty d\beta \int_0^\infty F(\xi,\beta)\,d\xi = \int_0^\infty \Psi(\infty,\beta)\,d\beta.$$

Hence

14) $$\int_0^\xi d\xi \int_0^\infty F(\xi,\beta)\,d\beta - \int_0^\infty d\beta \int_0^\infty F(\xi,\beta)\,d\xi$$
$$= -\int_0^\infty \{\Psi(\infty,\beta) - \Psi(\xi,\beta)\}\,d\beta.$$

We wish to show that this last integral approaches 0 as $\xi = \infty$.

Consider the function:

15) $$\varphi(\eta) = \int_\xi^\infty f(\xi,\eta)\cos\alpha\xi\,d\xi,$$

the lower limit of integration being merely a constant. The function $\varphi(\eta)$ is continuous and has a continuous derivative. For, the integral:

16) $$\int_0^\infty |f_\eta(\xi,\eta)|\,d\xi$$

converges uniformly by hypothesis, $0 \leqq \eta \leqq c$. Moreover, the function $f_\eta(\xi, \eta) \cos \alpha \xi$ is continuous. Hence the integral 15) can be differentiated by Leibniz's Rule.

Thus $\varphi(\eta)$ is seen to satisfy all the conditions of the Lemma of § 2, and so

$$17) \qquad \int_0^\infty d\beta \int_0^\infty \varphi(\eta) \cos \beta\eta \, d\eta = \frac{\pi}{2} \varphi(0^+).$$

Now,

$$\Psi(\infty, \beta) - \Psi(\xi, \beta) = \int_\xi^\infty F(\xi, \beta) \, d\xi = \int_0^\infty \varphi(\eta) \cos \beta\eta \, d\eta,$$

as follows at once from 4), 5) and 6). Hence

$$\int_0^\infty \{ \Psi(\infty, \beta) - \Psi(\xi, \beta) \} \, d\beta = \frac{\pi}{2} \int_\xi^\infty f(\xi, 0^+) \cos \alpha\xi \, d\xi.$$

Finally,

$$\left| \int_\xi^\infty f(\xi, 0^+) \cos \alpha \xi \, d\xi \right| \leqq \int_\xi^\infty |f(\xi, 0^+)| \, d\xi,$$

and so approaches 0 as $\xi = \infty$.

This completes the proof. We have used the hypotheses 5) and $4_1$), but not $4_2$). The latter are needed when the integrations with respect to $\alpha$, $\xi$ and $\beta$, $\eta$ are interchanged.

*The Case* $n \geqq 3$. The proof for the general case is given by a repeated application of the results above obtained. Consider the case $n = 3$. Here,

$$18) \; f(0^+, 0^+, 0^+) = \frac{8}{\pi^3} \int_0^\infty d\alpha \int_0^\infty d\xi \int_0^\infty d\beta \int_0^\infty d\eta \int_0^\infty d\gamma \int_0^\infty f(\xi, \eta, \zeta)$$

$$\cos \alpha\xi \, \cos \beta\eta \, \cos \gamma\zeta \, d\zeta.$$

The last two integrals, beginning on the right, can be replaced by $\frac{\pi}{2} f(\xi, \eta, 0^+)$. Hence the integrals with respect to $\xi$ and $\beta$ can be interchanged, provided the integrals

$$19)\quad \int_{-\infty}^{\infty} |f(x,y,z)|\,dx,\quad \int_{-\infty}^{\infty} |f(x,y,z)|\,dy,\quad \int_{-\infty}^{\infty} |f_y(x,y,z)|\,dx$$

converge uniformly, each in an arbitrary bounded interval, $z$ being merely any constant; and one of the integrals:

$$20)\qquad \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x,y,z)|\,dy,\qquad \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |f(x,y,z)|\,dx$$

converges.

We are thus led to the integral:

$$21)\qquad \int_0^{\infty} d\xi \int_0^{\infty} d\eta \int_0^{\infty} d\gamma \int_0^{\infty} f(\xi,\eta,\zeta) \cos\alpha\xi \cos\beta\eta \cos\gamma\zeta \, d\zeta.$$

Consider first the interchange of the integrations with respect to $\eta$ and $\gamma$. Here, the integrals

$$\int_{-\infty}^{\infty} |f(x,y,z)|\,dz,\quad \int_{-\infty}^{\infty} |f(x,y,z)|\,dy,\quad \int_{-\infty}^{\infty} |f_z(x,y,z)|\,dy$$

must converge uniformly in any finite interval, $x$ being merely any constant, and one of the integrals

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |f(x,y,z)|\,dz,\qquad \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} |f(x,y,z)|\,dy,$$

$x$ being any constant, must converge.

So now we come to the integral:

$$22)\quad \int_0^{\infty} d\xi \int_0^{\infty} d\gamma \int_0^{\infty} d\eta \int_0^{\infty} f(\xi,\eta,\zeta) \cos\alpha\xi \cos\beta\eta \cos\gamma\zeta \, d\zeta.$$

This integral will have the value

$$23)\qquad \int_0^{\infty} d\xi \int_0^{\infty} d\gamma \int_0^{\infty} \mathfrak{f}(\xi,\zeta) \cos\alpha\xi \cos\gamma\zeta \, d\zeta,$$

where

$$\mathfrak{f}(\xi,\zeta) = \int_0^{\infty} f(\xi,\eta,\zeta) \cos\beta\eta \, d\eta.$$

provided the order of the integrations with respect to $\eta$ and $\zeta$ in 22) can be reversed. This will be possible if one of the integrals:

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |f(x,y,z)|\, dz, \qquad \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} |f(x,y,z)|\, dy$$

converges.

We wish to reverse the order of the integrations with respect to $\xi$ and $\gamma$ in 23). Let

$$g(x,z) = \int_{-\infty}^{\infty} |f(x,y,z)|\, dy,$$

$$h(x,z) = \int_{-\infty}^{\infty} |f_z(x,y,z)|\, dy.$$

We need here to require the uniform convergence of the integrals:

$$\int_{-\infty}^{\infty} g(x,z)\, dz, \quad \int_{-\infty}^{\infty} g(x,z)\, dx, \quad \int_{-\infty}^{\infty} h(x,z)\, dx,$$

and the convergence of one of the integrals:

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} g(x,z)\, dz, \qquad \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} g(x,z)\, dx.$$

Thus we arrive finally at the integral:

$$24) \qquad \int_0^{\infty} d\gamma \int_0^{\infty} d\xi \int_0^{\infty} \mathfrak{f}(\xi,\zeta) \cos\alpha\xi \cos\gamma\zeta\, d\zeta.$$

Since an interchange of order in the integrations with respect to $\eta$ and $\zeta$ is allowable, we obtain as the final result:

$$25) \quad f(x,y,z) = \frac{1}{\pi^3} \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \int_0^{\infty} d\gamma \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} f(\xi,\eta,\zeta)$$

$$\cos\alpha(\xi - x) \cos\beta(\eta - y) \cos\gamma(\zeta - z)\, d\zeta.$$

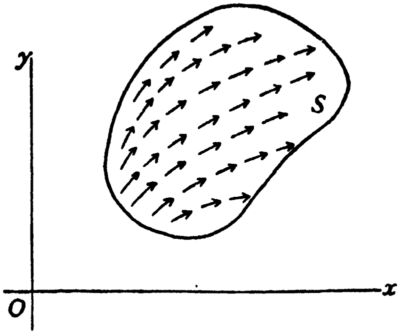
# Chapter XII

## Differential Equations. Existence Theorems

**§1. The Problem.** Consider the simplest case,

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

Let $f(x, y)$ be continuous in a region $S$ of the $(x, y)$-plane, the boundary points not being considered as belonging to the region. To each point $(x, y)$ of $S$ shall be assigned a direction, the slope of the line being defined as the value of $f(x, y)$. It is convenient to visualize these directions by means of short vectors drawn from the points. Thus we have spread out before us a two-dimensional family of these little vectors.



By a *solution* of the differential equation 1) is meant any function

$$y = \varphi(x) \tag{2}$$

such that the point $(x, y)$ lies in $S$ and the equation obtained by substituting the function 2) in 1) is identically true:

$$\varphi'(x) \equiv f[x, \varphi(x)]. \tag{3}$$

Thus $\varphi(x)$ must have a derivative, and from 3) this derivative must be continuous.

The graph of the solution 2) is a curve which courses $S$. At each one of its points it is tangent to the little vector pertaining to this point.

The question now arises, is there such a curve through each point of $S$, and is there more than one such curve? The answer to each question turns out to be affirmative. This gives us half of what we want — a solution through each point. The other half is just what we do not want — more than one solution.

For example, consider the differential equation:

$$4) \qquad \frac{dy}{dx} = 3y^{\frac{2}{3}}.$$

The function

$$5) \qquad y = \varphi(x, \alpha) = (x - \alpha)^3$$

is seen to be a solution; but it is not the only solution, for

$$6) \qquad y = \varphi(x) = 0$$

is also a solution; but the functions 5) and 6) do not exhaust the possibilities; for, through an arbitrary point of the plane there passes, not one, but an infinite number of solutions. Take the point (1, 1), say. Through this point passes the solution

$$y = x^3, \qquad 0 \leqq x;$$

but for $x < 0$ there is an arbitrary stretch* of the $x$-axis,

$$y = 0, \qquad \alpha < x < 0,$$

where $\alpha$ is any negative constant. And then, finally,

$$y = (x - \alpha)^3, \qquad x \leqq \alpha.$$

A similar situation exists for every point of the plane; and yet the function $f(x, y) = 3y^{\frac{2}{3}}$ is continuous without restriction, $S$ being here the entire plane.

We can forestall the occurrance of multiple solutions by imposing a further restriction on $f(x, y)$. It is sufficient to demand that $f$ possess a derivative with respect to $y$,

$$\frac{\partial f}{\partial y} = f_y(x, y),$$

and require that it be continuous** in $S$.

All that has been said can be extended at once to the general case of $n$ simultaneous differential equations of the first order in $n$

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* In particular, this stretch can extend to $-\infty$, or be replaced by a point, $\alpha = 0$.

** This condition can be lightened; it is enough that $f_y$ be bounded. On the other hand, still lighter conditions can be imposed (the so-called *Lipschitz* conditions). But in practice the condition of the text is fulfilled, if such a condition in any form is present, and so we choose the simpler requirement.

dependent variables; cf. § 3. Thus when $n = 2$ we have:

$$7) \qquad \frac{dy}{dx} = F(x, y, z), \qquad \frac{dz}{dx} = \Phi(x, y, z),$$

the functions $F$, $\Phi$ being continuous in a three-dimensional region $V$ of space, which shall not include any of its boundary points. To each point of $V$ shall be assigned a direction whose direction components $(1, F, \Phi)$ are suggested by writing 7) in the form:

$$8) \qquad \frac{dx}{1} = \frac{dy}{F} = \frac{dz}{\Phi}.$$

Thus we have before us a three-parameter family of little vectors. And now a solution of 7) will be given by two equations:

$$9) \qquad y = f(x), \qquad z = \varphi(x).$$

Geometrically, these equations represent a curve in space, coursing $V$, and such that, at each of its points, it is tangent to the little vector pertaining to this point.

**§ 2. The Existence Theorem.** THEOREM. *Consider the differential equation*

$$1) \qquad \frac{dy}{dx} = f(x, y),$$

*where $f(x, y)$ is continuous in an open region $S$ of the $(x, y)$-plane, and has a continuous derivative,*

$$\frac{\partial f}{\partial y} = f_y(x, y).$$

*Let $(x_0, y_0)$ be an arbitrary point of $S$. Then there exists a function,*

$$2) \qquad y = \varphi(x),$$

*having the following properties:*

i) *$\varphi(x)$ has a continuous derivative in the neighborhood of the point $x = x_0$ and takes on the value $y_0$ there:*

$$y_0 = \varphi(x_0);$$

ii) *$\varphi(x)$ satisfies the differential equation* 1) *throughout the neighborhood:*

$$3) \qquad \varphi'(x) = f[x, \varphi(x)];$$

iii) *The function $\varphi(x)$ is unique.*

The proof is given by the *Method of Successive Approximations.* Form the functions:

$$4)\qquad \begin{cases} y_1 = \int\limits_a^x f(x, b)\, dx + b, \\ y_2 = \int\limits_a^x f(x, y_1)\, dx + b, \\ \cdots\cdots\cdots\cdots \\ y_n = \int\limits_a^x f(x, y_{n-1})\, dx + b, \end{cases}$$

where $a = x_0$, $b = y_0$. Let

$$R: \qquad |x - a| \leqq A, \qquad |y - b| \leqq B$$

be a region lying in $S$, and let

$$5)\qquad |f(x,y)| \leqq M, \qquad |f_y(x,y)| \leqq G.$$

Then

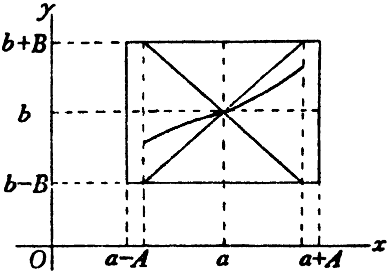
$$|y_1 - b| \leqq \int\limits_a^x |f(x, b)|\, dx \leqq M\,(x - a)$$

if $a \leqq x$, and in any case,

$$6)\qquad |y_1 - b| \leqq M\,|x - a|.$$

Now, if $y_1$ is to be available in the integral which defines $y_2$, then the point $(x, y_1)$ must lie in $S$. This will surely be the case if we restrict $x$ to the interval

$$7)\qquad a - h < x < a + h, \qquad 0 < h \leqq A,$$



choosing $h$ so that

$$8)\qquad Mh \leqq B.$$

Proceeding to $y_2$, we see in the same manner that

$$|y_2 - b| \leqq \int_a^x |f(x, y_1)|\, dx \leqq M(x - a)$$

when $a \leqq x < a + h$, and generally:

$$|y_2 - b| \leqq M\,|x - a|, \qquad |x - a| < h.$$

And so on:

$$|y_n - b| \leqq M\,|x - a|, \qquad |x - a| < h,$$

or

9) $$b - B < y_n < b + B.$$

Thus all of the functions $y_n$ are defined in the interval 7), the point $(x, y_n)$ lying in $S$. These shall be called by courtesy the *successive approximations*, and we will now show that they deserve their name. For, the function $y_n$ approaches a limit as $n$ becomes infinite. To prove this, let us write

10) $$y_n = b + (y_1 - y_0) + (y_2 - y_1) + \cdots + (y_n - y_{n-1})$$

and test the infinite series:

11) $$b + (y_1 - y_0) + (y_2 - y_1) + \cdots$$

for convergence. We have:

$$y_n - y_{n-1} = \int_a^x [f(x, y_{n-1}) - f(x, y_{n-2})]\, dx.$$

Now,

12) $$f(x, y_{n-1}) - f(x, y_{n-2}) = (y_{n-1} - y_{n-2}) f_y(x, Y),$$

where $Y$ lies between $y_{n-1}$ and $y_{n-2}$. Hence $(x, Y)$ is a point of $R$, and

13) $$|f_y(x, Y)| \leqq G.$$

We see, then, that

14) $$|y_n - y_{n-1}| \leqq \int_a^x G\,|y_{n-1} - y_{n-2}|\, dx$$

when $a \leqq x < a + h$, and a similar relation holds when $x$ is less than $a$.

Begin with $n = 2$. With the aid of 6) we have:

$$|y_2 - y_1| \leqq MG \int_a^x (x-a)\,dx = MG\frac{(x-a)^2}{2}$$

when $a \leqq x < a + h$, and generally:

15) $$|y_2 - y_1| \leqq MG\frac{|x-a|^2}{2!}, \qquad |x-a| < h.$$

Repeat the reasoning for $n = 3$; and so on. We arrive at the result:

16) $$|y_n - y_{n-1}| \leqq MG^{n-1}\frac{|x-a|^n}{n!}, \qquad |x-a| < h.$$

From 16) it follows that

17) $$|y_n - y_{n-1}| \leqq MG^{n-1}\frac{h^n}{n!}.$$

The right-hand side of this last inequality:

18) $$M_n = MG^{n-1}\frac{h^n}{n!},$$

is the general term of a convergent series, and hence the series 11) converges. But we can infer more than this. The quantity $M_n$ is independent of $x$. Hence the series 11) *converges uniformly* in the interval $|x - a| < h$, and since the terms of the series are continuous functions, the limiting function:

19) $$\varphi(x) = \lim_{n=\infty} y_n = b + (y_1 - y_0) + (y_2 - y_1) + \cdot \cdot \cdot$$

is continuous in this region.

Does $\varphi(x)$ have a derivative, and if so, does it have the value

$$\varphi'(x) = f[x, \varphi(x)]\ ?$$

The first question is answered in the affirmative by the general theorem for differentiating a series term-by-term, Chap. V, § 9, for all the conditions of that theorem are fulfilled. In particular,

20) $$\frac{d}{dx}(y_n - y_{n-1}) = f(x, y_{n-1}) - f(x, y_{n-2})$$

$$= (y_{n-1} - y_{n-2}) f_y(x, Y),$$

$$\left|\frac{d}{dx}(y_n - y_{n-1})\right| \leqq MG^{n-1}\frac{h^{n-1}}{(n-1)!}.$$

Thus the term-by-term derivative series converges uniformly, and its terms are continuous functions of $x$.

It appears, then, that each of the double limits:

$$\frac{d}{dx}\lim_{n=\infty} y_n, \qquad \lim_{n=\infty}\frac{dy_n}{dx}$$

converges, and that they are equal to $\varphi'(x)$, for this is precisely what the differentiability of the series term-by-term means. Hence, in particular

$$\varphi'(x) = \lim_{n=\infty} f(x, y_{n-1}) = f[x, \varphi(x)],$$

the last equation following from the fact that $f(x, y)$ is continuous in the point $x = x'$, $y = \varphi(x')$, and $y_{n-1}$ converges. Thus the answer to the second question is justified, too.

*Uniqueness.* It still remains to prove that the solution we have found by the foregoing method is *the only solution.* Let

21) $$u = \varphi(x),$$

and let

22) $$u + U = \Phi(x)$$

be any second solution. Then the two functions are identical in the whole interval $(a, a+h)$:

23) $$\Phi(x) - \varphi(x) \equiv 0, \qquad 0 \leqq x - a < h,$$

or

$$U \equiv 0.$$

If this were not the case, there would be a point $\xi$:

$$a \leqq \xi < a + h,$$

such that $U \equiv 0$ when $a \leqq x \leqq \xi$, but for any interval

$$\xi < x < \zeta < a + h,$$

however short, there will be points for which $U \neq 0$.

Plot the curve

$$y = |U|.$$

Let $\xi < x_1 < \zeta$ be a point for which the corresponding value $U_1$ of $U$ does not vanish:

$$0 < |U_1|.$$

Consider the roots of the equation

$$|U| = |U_1|$$

which lie in the above interval. There must be a smallest root, Chap. III, § 8, Ex. 6. Denote it by $\eta = \xi + \gamma$. Then

$$|U| < |U_1|, \qquad \xi \leqq x < \eta.$$

Now,

$$\frac{du}{dx} = f(x, u),$$

$$\frac{du}{dx} + \frac{dU}{dx} = f(x, u + U).$$

Hence

$$\frac{dU}{dx} = f(x, u + U) - f(x, u) = U f_y(x, Y),$$

where $Y$ lies between $u$ and $u + U$.

Integrating, we have:

$$U_1 = \int_\xi^\eta U f_y(x, Y)\, dx,$$

$$|U_1| \leqq |U_1| G\gamma,$$

$$1 \leqq G\gamma.$$

But $G > 0$, and $\gamma$ can be chosen arbitrarily small. From this contradiction follows the truth of the theorem.

Similar reasoning applies to the interval $(a - h, a)$; or this case can be referred to the one just treated by a transformation, $x' = -x$.

## EXERCISES

1. The solution 2) of the differential equation 1) is a function not only of $x$ but also of $(x_0, y_0)$. Holding $x_0 = a$ fast and letting $y_0 = b$ vary, show that the solution,

$$y = \varphi(x, b),$$

is a continuous function of the two independent variables $(x, b)$.

2. Let $f(x, y, \alpha)$ be continuous, together with $f_y(x, y, \alpha)$, in the neighborhood of a point $(x_0, y_0, \alpha_0)$. Show that the solution of the differential equation

$$\frac{dy}{dx} = f(x, y, \alpha),$$

which passes through the point $(x_0, y_0)$:

$$y = \varphi(x, \alpha),$$

is a continuous functions of $(x, \alpha)$ in the neighborhood of the point $(x_0, \alpha_0)$.

Hold $x_0$ fast, but let $y_0 = b$ be a continuous function of $\alpha$.

3. If to the hypotheses of Question 2 are added the existence and continuity of the derivatives*

$$\frac{\partial^2 f}{\partial y^2} = f_{yy}, \qquad \frac{\partial^2 f}{\partial y \partial \alpha} = f_{y\alpha},$$

and also of $b = b(\alpha)$, show that the function $\varphi(x, \alpha)$ admits a derivative with respect to $\alpha$, continuous in the neighborhood of $(x_0, \alpha_0)$, and that the same is true of the function $\varphi'(x, \alpha)$.

4. Given the differential equation

$$\frac{dy}{dx} = xe^y - ye^{-x} + \sin x + \log(1 + y^2).$$

Show that it has a unique solution passing through an arbitrary point of the plane:

$$y = \varphi(x;\, x_0, y_0), \qquad -\infty < x < \infty;$$

and that the corresponding curves sweep out the entire $(x, y)$-plane just once.

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* Cotton has given a more general theorem, the derivatives of the second order not being required to exist; cf. § 4 and Goursat, *Cours d'analyse mathématique*, vol. III (1915) § 460.

**§3. Continuation; $n$ Equations.** THEOREM. *Consider the simultaneous system of differential equations:*

$$1)\qquad \frac{dy_k}{dx}=f_k(x,y_1,\cdots,y_n), \qquad k=1,\cdots,n,$$

*where $f_k(x,y_1,\cdots,y_n)$ is continuous, together with its derivatives of the first order with respect to $y_1,\cdots,y_n$, in the neighborhood of a point $(a,b_1,\cdots,b_n)$. Then there exist $n$ functions:*

$$2)\qquad y_1=\varphi_1(x), \quad \cdots, \quad y_n=\varphi_n(x)$$

*having the following properties:*

i) *$\varphi_k(x)$ has a continuous derivative in the neighborhood of the point $x=a$ and*

$$\varphi_k(a)=b_k;$$

ii) *These functions satisfy the given system of differential equations:*

$$\varphi_k'(x)\equiv f_k[x,\varphi_1(x),\cdots,\varphi_n(x)], \qquad k=1,\cdots,n;$$

iii) *The functions 2) are unique.*

The proof given in § 2 for the case $n=1$ is applicable in substance, the requisite extensions being adequately indicated in the case $n=2$, to which we now turn. Let us write the system 1) in the form:

$$3)\qquad \frac{dy}{dx}=F(x,y,z), \qquad \frac{dz}{dx}=\Phi(x,y,z),$$

the functions being continuous, together with their first derivatives with respect to $y$, $z$, in the region

$$R:\quad |x-a|\leqq A, \qquad |y-b|\leqq B, \qquad |z-c|\leqq C.$$

Moreover, let

$$4)\qquad \begin{cases} |F(x,y,z)|\leqq M, & |\Phi(x,y,z)|\leqq M; \\ |F_y(x,y,z)|\leqq G, \quad \cdots, & |\Phi_z(x,y,z)|\leqq G. \end{cases}$$

Form the approximations:

$$5)\quad \begin{cases} y_1 = \displaystyle\int_a^x F(x, b, c)\, dx + b, & z_1 = \displaystyle\int_a^x \Phi(x, b, c)\, dx + c; \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ y_n = \displaystyle\int_a^x F(x, y_{n-1}, z_{n-1})\, dx + b, & \\ & z_n = \displaystyle\int_a^x \Phi(x, y_{n-1}, z_{n-1})\, dx + c, \end{cases}$$

restricting $x$ to an interval

$$6)\qquad -h < x < h, \qquad 0 < h \leqq A,$$

where $h$ is so chosen that

$$7)\qquad Mh \leqq B, \qquad Mh \leqq C.$$

To examine the limits: $\lim y_n$ and $\lim z_n$, write

$$y_n = b + (y_1 - y_0) + (y_2 - y_1) + \cdots + (y_n - y_{n-1}),$$

$$z_n = c + (z_1 - z_0) + (z_2 - z_1) + \cdots + (z_n - z_{n-1}),$$

where $y_0 = b$, $z_0 = c$, and study the infinite series:

$$8)\qquad b + (y_1 - y_0) + (y_2 - y_1) + \cdots,$$

$$9)\qquad c + (z_1 - z_0) + (z_2 - z_1) + \cdots.$$

We have from 5) the appraisals:

$$10)\quad |y_1 - b| \leqq M|x - a|, \qquad |z_1 - c| \leqq M|x - a|.$$

Furthermore,

$$y_n - y_{n-1} = \int_a^x \{ F(x, y_{n-1}, z_{n-1}) - F(x, y_{n-2}, z_{n-2}) \}\, dx.$$

By the Law of the Mean:

$$F(x, y_{n-1}, z_{n-1}) - F(x, y_{n-2}, z_{n-2}) =$$

$$(y_{n-1} - y_{n-2}) F_y(x, Y, Z) + (z_{n-1} - z_{n-2}) F_z(x, Y, Z).$$

By virtue of 4):

11) $$|F(x, y_{n-1}, z_{n-1}) - F(x, y_{n-2}, z_{n-2})| \leqq G\{|y_{n-1} - y_{n-2}| + |z_{n-1} - z_{n-2}|\}.$$

Hence

12) $$|y_n - y_{n-1}| \leqq \int_a^x G\{|y_{n-1} - y_{n-2}| + |z_{n-1} - z_{n-2}|\}\,dx;$$

and similarly:

13) $$|z_n - z_{n-1}| \leqq \int_a^x G\{|y_{n-1} - y_{n-2}| + |z_{n-1} - z_{n-2}|\}\,dx,$$

where $a \leqq x < a + h$, with a similar relation if $a - h < x < a$.

For $n = 1$, these appraisals are replaced by 10). When $n = 2$ they yield with the aid of 10):

14) $$|y_2 - y_1| \leqq 2MG\frac{(x-a)^2}{2!}, \quad |z_2 - z_1| \leqq 2MG\frac{(x-a)^2}{2!}.$$

And now it is shown by mathematical induction that, generally,

15) $$\begin{cases} |y_n - y_{n-1}| \leqq M(2G)^{n-1}\dfrac{|x-a|^n}{n!}, \\ |z_n - z_{n-1}| \leqq M(2G)^{n-1}\dfrac{|x-a|^n}{n!}, \end{cases}$$

where $x$ lies in 6). In particular, then :

16) $$|y_n - y_{n-1}| \leqq M_n, \qquad |z_n - z_{n-1}| \leqq M_n,$$

where

$$M_n = M(2G)^{n-1}\frac{h^n}{n!}$$

The series

$$\sum_n M_n$$

converges. Hence the series 8) and 9) converge. Moreover, their terms are continuous and the series converge uniformly. By Chap. V, § 4 they define functions:

17)
$$\begin{cases} \varphi(x) = b + (y_1 - y_0) + (y_2 - y_1) + \cdots, \\ \psi(x) = c + (z_1 - z_0) + (z_2 - z_1) + \cdots, \end{cases}$$

continuous in the interval 6) and such that

$$|\varphi(x) - b| \leqq B, \qquad |\psi(x) - c| \leqq C.$$

These are the limits sought:

18)
$$\lim_{n=\infty} y_n = \varphi(x), \qquad \lim_{n=\infty} z_n = \psi(x).$$

It remains to show that these functions have derivatives and satisfy the given system of differential equations 3).

The series 8) and 9) can be differentiated term-by-term. For

$$\frac{d}{dx}(y_n - y_{n-1}) = F(x, y_{n-1}, z_{n-1}) - F(x, y_{n-2}, z_{n-2}).$$

From 11) and 16) it follows that

$$\left|\frac{d}{dx}(y_n - y_{n-1})\right| \leqq 2GM_{n-1}.$$

Hence all the hypotheses of the theorem of Chap. V, § 9 are fulfilled, and so $\varphi(x)$ admits a derivative:

$$\varphi'(x) = \sum_n \{ F(x, y_{n-1}, z_{n-1}) - F(x, y_{n-2}, z_{n-2}) \}$$

$$= \lim_{n=\infty} F(x, y_n, z_n) = F[x, \varphi(x), \psi(x)].$$

Similarly,

$$\psi'(x) = \Phi[x, \varphi(x), \psi(x)].$$

Thus a solution,

19)
$$y = \varphi(x), \qquad z = \psi(x)$$

of the given system of differential equations 3) is obtained.

It remains to prove the uniqueness. Let

$$u = \varphi(x), \qquad v = \psi(x),$$

and let

$$u + U = \Phi(x), \qquad v + V = \Psi(x)$$

be any second solution. Then

$$\frac{dU}{dx} = F(x, u + U, v + V) - F(x, u, v)$$

$$= U F_y(x, Y, Z) + V F_z(x, Y, Z),$$

and similarly:

$$\frac{dV}{dx} = U \Phi_y(x, Y, Z) + V \Phi_z(x, Y, Z).$$

We wish to show that $U$, $V$ vanish identically. If they do not, suppose that

$$\omega(x) = |U| + |V| = 0, \qquad a \leqq x \leqq \xi < a + h;$$

$$\omega(x_1) \neq 0, \quad \xi < x_1,$$

where $x_1$ can be chosen arbitrarily near to $\xi$. Let $\eta = \xi + \gamma$ be the smallest root of the equation

$$\omega(x) = \omega(x_1).$$

Let

$$U(\eta) = U_1, \qquad V(\eta) = V_1.$$

Then

$$U_1 = \int_\xi^\eta \{ U F_y + V F_z \} \, dx,$$

$$|U_1| \leqq G\{|U_1| + |V_1|\}\gamma;$$

and similarly,

$$|V_1| \leqq G\{|U_1| + |V_1|\}\gamma.$$

Hence

$$\{|U_1| + |V_1|\} \leqq 2G\{|U_1| + |V_1|\}\gamma,$$

or

$$1 \leqq 2G\gamma.$$

Since $\gamma$ can be made arbitrarily small, the proof of uniqueness is complete.

### EXERCISES

1. Show that, under the hypotheses of the Theorem, when $n = 2$, there exists a sub-region

$$R': \qquad |x-a| \leqq A', \quad |y-b| \leqq B', \quad |z-c| \leqq C',$$

and a positive constant $h'$, such that, if $(x_0, y_0, z_0)$ is an arbitrary point of $R'$, there is one and only one solution of the differential equations 3):

$$y = f(x;\, x_0, y_0, z_0), \qquad z = \varphi(x;\, x_0, y_0, z_0),$$

defined throughout the interval

$$-h' < x - x_0 < h',$$

with $f(x_0;\, x_0, y_0, z_0) = y_0$ and $\varphi(x_0;\, x_0, y_0, z_0) = z_0$.

Generalize to the case $n = n$.

2. Carry through the detailed proof in the general case, $n = n$.

3. Extend the Exercises of § 2 to the general case of this paragraph.

4. Let the function $f_k(x, y_1, \cdots, y_n)$ of the system of differential equations 1) be continuous when $x$ lies in the interval

$$a \leqq x \leqq b$$

and $y_1, \cdots, y_n$ are unrestricted, and let the first derivatives of $f_k$ with respect to the $y$'s be bounded. Show that through an arbitrary point $(x_0, y_1^0, \cdots, y_n^0)$ passes one and only one solution 2), defined for the whole interval $(a, b)$.

**§ 4. The Semi-Linear Case.** Let the simultaneous system:[*]

$$1) \qquad \begin{cases} \dfrac{dy}{dx} = f(x, y) \\[2ex] \dfrac{dz}{dx} = z\,\varphi(x, y) + \psi(x, y) \end{cases}$$

be given, where $f(x, y)$, $\varphi(x, y)$, $\psi(x, y)$ are continuous in the neighborhood of a point $(a, b)$, and where, moreover,

$$\frac{\partial f}{\partial y} = f_y(x, y)$$

---

[*] cf. foot-note, p. 356.

exists and is also continuous there. The first equation admits a unique solution,

$$2)\qquad Y = \varphi(x), \qquad \varphi(a) = b,$$

in the neighborhood of the point $x = a$,

$$3)\qquad a - h < x < a + h.$$

When this function is substituted for $y$ in the second equation, the value of $h$ being conceivably cut down, so as to bring the point $(x, Y)$ within the above neighborhood, the latter becomes a linear differential equation in $z$, and it admits, for an arbitrary initial value, $z = c$, a unique solution $Z$, defined throughout the whole interval 3). Thus

$$4)\qquad \begin{cases} \dfrac{dY}{dx} = f(x, Y), \\[2ex] \dfrac{dZ}{dx} = Z\varphi(x, Y) + \psi(x, Y). \end{cases}$$

So far, then, as the solution of the simultaneous system of differential equations 1) is concerned, there is nothing left to be desired. By means of the resulting functions, which satisfy Equations 4), we proceed to develop a

Convergence Theorem. *Let a sequence of successive approximations be defined as in* §§ 2, 3:

$$5)\qquad \begin{cases} y_n = \displaystyle\int_a^x f(x, y_{n-1})\,dx + b, \\[3ex] z_n = \displaystyle\int_a^x \{ z_{n-1}\,\varphi(x, y_{n-1}) + \psi(x, y_{n-1}) \}\,dx + c, \end{cases}$$

*with* $y_0 = b$ *and* $z_0 = \gamma$, *where* $\gamma$ *is arbitrary. Then*

$$\lim_{n=\infty} y_n = Y(x), \qquad \lim_{n=\infty} z_n = Z(x).$$

*Moreover,* $y_n = y_n(x)$ *and* $z_n = z_n(x)$ *converge uniformly in the interval* 3).

So far as $\lim y_n$ is concerned, the proof comes under the earlier case. As regards $\lim z_n$, write from 4):

$$Z = \int_a^x \{ Z\varphi(x, Y) + \psi(x, Y) \} dx + c,$$

and form the difference:

6)
$$Z - z_n = \int_a^x \{ Z\varphi(x, Y) - z_{n-1}\varphi(x, y_{n-1}) + \psi(x, Y) - \psi(x, y_{n-1}) \} dx.$$

Set:

$$\begin{cases} \varphi(x, y_n) = \varphi(x, Y) + \zeta_n, \\ \psi(x, y_n) = \psi(x, Y) + \zeta_n'. \end{cases}$$

Since $y_n(x)$ converges uniformly in the interval 3), we have:

$$|\zeta_n| < \eta, \qquad |\zeta_n'| < \eta, \qquad m \leqq n,$$

where $\eta$ is an arbitrary positive constant, and $m$ is independent of $x$. Rewrite 6) in the form:

7) $$Z - z_n = \int_a^x \{ \varphi(x, y_{n-1})(Z - z_{n-1}) - Z\zeta_{n-1} - \zeta_{n-1}' \} dx.$$

Let $\epsilon$ be an arbitrary positive number. Then $\eta$ can be so chosen that

8) $$| - Z\zeta_{n-1} - \zeta_{n-1}' | < \epsilon, \qquad m \leqq n,$$

where $x$ lies in 3) and $m$ is independent of $x$. Moreover,

$$|\varphi(x, y_{n-1})| \leqq G,$$

where $G$ is a suitable positive constant. Hence

9) $$|Z - z_n| \leqq \int_a^x \{ G|Z - z_{n-1}| + \epsilon \} dx, \qquad m \leqq n,$$

$$a \leqq x < a + h.$$

Let $M$ be so chosen that

$$G\,|\,Z - z_{m-1}\,| + \epsilon \leqq M$$

when $x$ is in 3). Then

$$|\,Z - z_m\,| \leqq M\,|\,x - a\,|.$$

From 9) we now infer that

$$|\,Z - z_{m+1}\,| \leqq MG\frac{|\,x - a\,|^2}{2!} + \epsilon\,|\,x - a\,|.$$

By mathematical induction it follows that

$$10)\qquad |\,Z - z_{m+r}\,| \leqq MG^r\frac{|\,x - a\,|^{r+1}}{(r+1)!} + \epsilon\sum_{k=1}^{r}\frac{|\,x - a\,|^k}{k!}.$$

Hence we arrive at the final appraisal:

$$11)\qquad |\,Z - z_{m+r}\,| \leqq MG^r\frac{h^{r+1}}{(r+1)!} + \epsilon\,(e^h - 1),$$

and the Convergence Theorem is established.

Both theorem and proof admit an immediate generalization:—

CONVERGENCE THEOREM. GENERAL CASE. *Let the system of simultaneous differential equations be given:*

$$12)\qquad \begin{cases} \dfrac{dy_k}{dx} = f_k(x, y_1, \cdots, y_m), \qquad k = 1, \cdots, m; \\[2ex] \dfrac{dz_j}{dx} = \displaystyle\sum_{i=1}^{l} z_i\,\varphi_{ij}(x, y_1, \cdots, y_m) + \varphi_j(x, y_1, \cdots, y_m), \\[1ex] \qquad\qquad\qquad j = 1, \cdots, l, \end{cases}$$

*where $f_k$, $\varphi_{ij}$, $\varphi_j$ are continuous throughout the neighborhood of a point $(a, b_1, \cdots, b_m)$, and where, furthermore, the derivatives*

$$\frac{\partial f_k}{\partial y_\alpha}, \qquad k, \alpha = 1, \cdots, m,$$

*exist and are also continuous there.*

*Form the successive approximations:*

$$13)\quad \begin{cases} y_k^{(n)} = \displaystyle\int_{x_0}^{x} f_k(x, y_1^{(n-1)}, \cdots, y_m^{(n-1)})\, dx + y_k^0, \\ z_j^{(n)} = \displaystyle\int_{x_0}^{x} \Big( \sum_{i=1}^{l} z_i^{(n-1)} \varphi_{ij}(x, y_1^{(n-1)}, \cdots, y_m^{n-1}) \\ \qquad\qquad + \varphi_j(x, y_1^{(n-1)}, \cdots, y_m^{(n-1)}) \Big) dx + z_j^0, \end{cases}$$

$$k = 1, \cdots, m; \qquad i, j = 1, \cdots, l; \qquad 1 < n,$$

*where* $(x_0, y_1^0, \cdots, y_m^0)$ *is an arbitrary point in the neighborhood of* $(a, b_1, \cdots, b_m)$, *and* $z_j^0$ *is wholly unrestricted.*

*For* $n = 1$, *replace* $y_k^{(n-1)}$, $z_i^{(n-1)}$ *by* $\beta_k$, $\gamma_i$, *where* $(\beta_1, \cdots, \beta_m)$ *lies in the neighborhood of* $(b_1, \cdots, b_m)$, *and* $(\gamma_1, \cdots, \gamma_l)$ *is wholly unrestricted. Let* $(c_1, \cdots, c_l)$ *be an arbitrary fixed point.*

*Then there is an interval:*

$$a - h < x < a + h,$$

*and a neighborhood of the point*

$$(x_0, y_1^0, \cdots, y_m^0, z_1^0, \cdots, z_l^0) = (a, b_1, \cdots, b_m, c_1, \cdots, c_l)$$

*within which the functions:*

$$\begin{cases} y_k^{(n)} = y_k^{(n)}(x; x_0, y_1^0, \cdots, y_m^0), & k = 1, \cdots, m; \\ z_j^{(n)} = z_j^{(n)}(x; x_0, y_1^0, \cdots, y_m^0, z_1^0, \cdots, z_l^0) & j = 1, \cdots, l, \end{cases}$$

*converge uniformly to the solution of Equations* 12)*:*

$$14)\quad \begin{cases} Y_k = \theta_k(x; x_0, y_1^0, \cdots, y_m^0), & k = 1, \cdots, m; \\ Z_j = \varpi_j(x; x_0, y_1^0, \cdots, y_m^0, z_1^0, \cdots, z_l^0), & j = 1, \cdots, l. \end{cases}$$

Finally, the differential equations themselves may depend on parameters. The simplest case,

$$15)\quad \begin{cases} \dfrac{dy}{dx} = f(x, y, \alpha), \\ \dfrac{dz}{dx} = z\varphi(x, y, \alpha) + \psi(x, y, \alpha), \end{cases}$$

is illustrative for the general case. Let $f(x, y, \alpha)$, $\varphi(x, y, \alpha)$, $\psi(x, y, \alpha)$ be continuous in the neighborhood of $(a, b, \alpha')$. Let

$$\frac{\partial f}{\partial y} = f_y(x, y, \alpha)$$

exist and be continuous there, also.

Convergence Theorem. *Let a sequence of successive approximations be defined by the formulas:*

$$16)\qquad \begin{cases} y_n = \displaystyle\int_{x_0}^{x} f(x, y_{n-1}, \alpha)\, dx + y_0, \\ z_n = \displaystyle\int_{x_0}^{x} z_{n-1} f_y(x, y_{n-1}, \alpha)\, dx + z_0, \end{cases}$$

$1 < n$, *where* $(x_0, y_0, z_C, \alpha)$ *is an arbitrary point of a certain neighborhood of* $(a, b, c, \alpha')$; *here,* $c$ *is an arbitrary constant. When* $n = 1$, *the arguments* $y_{n-1}$, $z_{n-1}$ *shall be replaced by* $\beta$, $\gamma$, *where* $\beta$ *lies in the neighborhood of* $b$, *and* $\gamma$ *is an arbitrary constant. Then there is an interval:*

$$a - h < x < a + h,$$

*and a neighborhood of the point* $(x_0, y_0, z_0, \alpha) = (a, b, c, \alpha')$, *within which the functions*

$$y_n = y_n(x;\, x_0, y_0, z_0, \alpha), \qquad z_n = z_n(x;\, x_0, y_0, z_0, \alpha)$$

*converge uniformly for* $n = \infty$ *to the solution of Equations* 15)

$$17)\qquad \begin{cases} Y = \varphi(x;\, x_0, y_0, z_0, \alpha), \\ Z = \psi(x;\, x_0, y_0, z_0, \alpha). \end{cases}$$

**§5. Dependence on Parameters.** Beginning with the simplest case,

$$1)\qquad \frac{dy}{dx} = f(x, y),$$

we have the solution:

$$2)\qquad y = \varphi(x;\, x_0, y_0),$$

where $\varphi(x;\, x_0, y_0)$ is continuous in the neighborhood of $(a;\, a, b)$, and where 2) holds uniformly for an interval

3) $$a - h < x < a + h,$$

$h$ being constant for points $(x_0, y_0)$ lying in a certain neighborhood of the point $(a, b)$.

So much can be inferred directly from the proof of existence by successive approximations. That process did not throw light, however, on the existence of the derivatives

$$\frac{\partial \varphi}{\partial x_0}, \qquad \frac{\partial \varphi}{\partial y_0}.$$

By means of the results of § 4 the existence and continuity of these derivatives can be established under no additional hypotheses concerning $f(x, y)$ beyond those of § 2, Theorem.

Start with the sequence of successive approximations:

4) $$y_n = \int_{x_0}^{x} f(x, y_{n-1})\, dx + y_0.$$

Then

5) $$y_n = \varphi_n(x;\, x_0, y_0)$$

admits continuous first partial derivatives with respect to $x_0, y_0$. We wish to show that the limiting function:

6) $$Y = \lim_{n=\infty} y_n = y_0 + (y_1 - y_0) + (y_2 - y_1) + \cdots,$$

admits continuous first partial derivatives with respect to $x_0, y_0$. This will be the case if the conditions of the Theorem of Chap. V, § 9 are fulfilled. The one outstanding hypothesis is that of uniform convergence of the series of derivatives:

7) $$\frac{\partial y_0}{\partial x_0} + \sum_{n=1}^{\infty} \frac{\partial (y_n - y_{n-1})}{\partial x_0}, \qquad \frac{\partial y_0}{\partial y_0} + \sum_{n=1}^{\infty} \frac{\partial (y_n - y_{n-1})}{\partial y_0}.$$

That this hypothesis is in fact fulfilled, can be shown as follows.

Suppose for the moment that these derivatives do exist, and write them:

8) $$z = \frac{\partial \varphi}{\partial x_0}, \qquad u = \frac{\partial \varphi}{\partial y_0}.$$

Now

$$Y = \int_{x_0}^{x} f(x, Y)\,dx + y_0.$$

Hence

$$9)\qquad \begin{cases} \dfrac{\partial \varphi}{\partial x_0} = \displaystyle\int_{x_0}^{x} \dfrac{\partial \varphi}{\partial x_0} f_y(x, Y)\,dx - f(x_0, y_0) \\[2ex] \dfrac{\partial \varphi}{\partial y_0} = \displaystyle\int_{x_0}^{x} \dfrac{\partial \varphi}{\partial y_0} f_y(x, Y)\,dx + 1. \end{cases}$$

From the foregoing reasoning we infer the following: — A necessary condition for the existence of the function 2) and its partial derivatives 8) is that these functions satisfy the system of simultaneous semi-linear differential equations:

$$10)\qquad \begin{cases} \dfrac{dy}{dx} = f(x, y), \\[2ex] \dfrac{dz}{dx} = z f_y(x, y), \\[2ex] \dfrac{du}{dx} = u f_y(x, y), \end{cases}$$

with the initial conditions:

$$11)\qquad x = x_0, \qquad y = y_0, \qquad z_0 = -f(x_0, y_0), \qquad u_0 = 1.$$

The form of this result, combined with the developments of § 4, suggests a means of establishing the uniform convergence of the series 7) and thus completing our proof. By § 4 the system 10) can be solved by the aid of the successive approximations:

$$12)\qquad \begin{cases} y_n = \displaystyle\int_{x_0}^{x} f(x, y_{n-1})\,dx + y_0, \\[2ex] z_n = \displaystyle\int_{x_0}^{x} z_{n-1} f_y(x, y_{n-1})\,dx - f(x_0, y_0), \\[2ex] u_n = \displaystyle\int_{x_0}^{x} u_{n-1} f_y(x, y_{n-1})\,dx + 1, \end{cases}$$

$1 < n$. When $n = 1$, the definition of $y_1$, $z_1$, $u_1$ can be made to serve the application we are about to consider.

Turning to Equation 4) we see that

$$13)\quad \begin{cases} \dfrac{\partial y_n}{\partial x_0} = \displaystyle\int_{x_0}^{x} \dfrac{\partial y_{n-1}}{\partial x_0} f_y(x, y_{n-1})\, dx - f(x_0, y_0), \\[2ex] \dfrac{\partial y_n}{\partial y_0} = \displaystyle\int_{x_0}^{x} \dfrac{\partial y_{n-1}}{\partial y_0} f_y(x, y_{n-1})\, dx + 1. \end{cases}$$

When $n = 1$, these equations become:

$$14)\quad \frac{\partial y_1}{\partial x_0} = -f(x_0, y_0), \qquad \frac{\partial y_1}{\partial y_0} = \int_{x_0}^{x} f_y(x, y_0)\, dx + 1.$$

Returning to Equations 12) let us agree that, when $n = 1$, the arguments $y_{n-1}$, $z_{n-1}$, $u_{n-1}$ shall be set equal to $y_0$, 0, 1 respectively. Then, by virtue of 14):

$$y_1 = \int_{x_0}^{x} f(x, y_0)\, dx + y_0, \qquad z_1 = \frac{\partial y_1}{\partial x_0}, \qquad u_1 = \frac{\partial y_1}{\partial y_0}.$$

We now subtract Equations 12) from Equations 13):

$$15)\quad \begin{cases} \dfrac{\partial y_n}{\partial x_0} - z_n = \displaystyle\int_{x_0}^{x} f_y(x, y_{n-1}) \left[ \dfrac{\partial y_{n-1}}{\partial x_0} - z_{n-1} \right] dx, \\[2ex] \dfrac{\partial y_n}{\partial y_0} - u_n = \displaystyle\int_{x_0}^{x} f_y(x, y_{n-1}) \left[ \dfrac{\partial y_{n-1}}{\partial y_0} - u_{n-1} \right] dx, \end{cases}$$

$1 < n$. Hence it appears that identically, for all such values of $n$:

$$\frac{\partial y_n}{\partial x_0} = z_n, \qquad \frac{\partial y_n}{\partial y_0} = u_n,$$

and thus the uniform convergence of the series 7) is established.

The extension of the result to the case that the differential equation depends on parameters presents no difficulty, and we are thus led to the general theorem: —

THEOREM. *Consider the system of differential equations:*

$$\frac{dy_k}{dx}=f_k(x;\, y_1, \cdots, y_n;\, \lambda_1, \cdots, \lambda_m),$$

$k = 1, \cdots, n$, *where* $f_k$ *together with the first partial derivatives:*

$$\frac{\partial f_k}{\partial y_j}, \qquad \frac{\partial f_k}{\partial \lambda_\alpha},$$

*is continuous in the neighborhood of the point* $(a, b_1, \cdots, b_n, \lambda_1', \cdots, \lambda_m')$. *Then the solution:*

$$y_k = \varphi_k(x;\, x_0, y_1^0, \cdots, y_n^0;\, \lambda_1, \cdots, \lambda_m),$$

*where*

$$y_k^0 = \varphi_k(x_0;\, x_0, y_1^0, \cdots, y_n^0;\, \lambda_1, \cdots, \lambda_m),$$

*is continuous together with all* $n + m + 2$ *first partial derivatives, in the neighborhood of the point* $(a;\, a, b_1, \cdots, b_n;\, \lambda_1', \cdots, \lambda_m')$.

Since

$$\frac{d\varphi_k}{dx}=f_k(x;\, \varphi_1, \cdots, \varphi_n;\, \lambda_1, \cdots, \lambda_m),$$

it is clear that each of the derivatives

$$\frac{\partial}{\partial y_j^0}\frac{d\varphi_k}{dx}, \qquad \frac{\partial}{\partial \lambda_\alpha}\frac{d\varphi_k}{dx}$$

exists and is continuous.

## § 6. Implicit Integral Relations.

The differential equation of § 2:

$$\frac{dy}{dx}=f(x, y),$$

admits the solution:

$$y = \varphi(x;\, x_0, y_0), \tag{1}$$

where $\varphi$ is continuous together with its derivatives of the first order in the neighborhood of the point $(a, a, b)$. It might seem, then, that we had found in 1) a primitive (= solution) of the differential equation which depends on *two* arbitrary constants, $x_0$ and $y_0$. But in substance there is only one; for if we hold $x_0$ fast, we obtain the whole family by allowing $y_0$ alone to vary. If, then, we write, as in § 2, Ex. 1, the solution in the form:

2) $$y = \varphi(x, b),$$

we have what may be called the "general" solution, depending on a single arbitrary constant.

In fact, it is through the present existence theorems that one can attach a precise meaning to the term *general solution.* The formalism of special differential equations, integrated by special devices, is a morass of half-true theorems, which are unreliable in any given case. Take, for example, the differential equation :

$$\frac{dy}{dx} = y.$$

The function

$$y = e^{x+c}$$

is a solution of the form

$$y = \varphi(x, c) ;$$

i.e. containing an arbitrary constant — and that is all that is called for in many of the books on elementary differential equations, such as are used in a sophomore or junior course. But this solution is not general, for there is no value of $c$ which gives the solution

$$y = 0.$$

On the other hand, the solution

$$y = Ce^x$$

*is* general. But how is one to know, from the formalism of elementary functions, even now, whether all solutions have been rounded up? In this case, the existence theorem of § 2 is unnecessary, as the explicit function supplies the demand; but the *uniqueness theorem* is essential, and answers the question with which formalism is powerless to deal.

The example of § 1:

$$\frac{dy}{dx} = 3y^{\frac{2}{3}},$$

$$y = \varphi(x, \alpha) = (x - \alpha)^3,$$

is also a case in point.

*Implicit Form of the Integral.* Consider the solution 1) of this paragraph. Let $(x_1, y_1)$ be any point on this curve, in a suitable neighborhood of $(x_0, y_0)$. Then

$$y_1 = \varphi(x_1; x_0, y_0).$$

On the other hand, apply the existence theorem to the given differential equation 1) of § 2, considered for the neighborhood of $(x_1, y_1)$. Then the solution is given by the equation:

$$y = \varphi(x; x_1, y_1),$$

where $\varphi$ is the same function as before. In particular, this curve goes through the point $(x_0, y_0)$ because of the theorem of § 3, Ex. 1 and the uniqueness theorem:

$$y_0 = \varphi(x_0; x_1, y_1).$$

But $(x_1, y_1)$ was any point on the curve 1). Hence

$$y = \varphi(x; x_0, y_0) \qquad \text{and} \qquad y_0 = \varphi(x_0; x, y)$$

are equivalent equations.

The result can be extended at once to the general case of § 3. Let the solution 2) be written in the form:

$$3) \qquad y_k = \varphi_k(x; x_0, y_1^0, \cdots, y_n^0), \qquad k = 1, \cdots, n.$$

If $x_0 = a$ is held fast and $y_k^0 = b_k$ is regarded as a parameter, then

$$4) \qquad y_k = \varphi_k(x; a, b_1, \cdots, b_n), \qquad k = 1, \cdots, n,$$

is the *general solution*, in that it gives *every* solution which courses the neighborhood of the point $(x_0, y_1^0, \cdots, y_n^0)$. The solution depends on $n$ arbitrary constants, the $b_1, \cdots, b_n$.

On the other hand, the points $(x_0, y_1^0, \cdots, y_n^0)$ and $(x, y_1, \cdots, y_n)$ can be interchanged as above, and we have as the equivalent of 3) the $n$ implicit equations:

$$5) \qquad y_k^0 = \varphi_k(x_0; x, y_1, \cdots, y_n), \qquad k = 1, \cdots, n.$$

Observe that the Jacobian of these $n$ functions:

$$6)\qquad \frac{\partial(\varphi_1, \cdots, \varphi_n)}{\partial(y_1, \cdots, y_n)},$$

has the value 1 in the point $x = x_0$, $y_k = y_k^0$. For, in 3), $\varphi_k$ reduces to $y_k^0$ when $x = x_0$. That the derivatives involved exist and are continuous, has been shown in § 5.

A more general definition of an *integral* of the given system of differential equations is the following. Let

$$\varpi(x, y_1, \cdots, y_n) \not\equiv \text{const.}$$

be continuous, together with its first derivatives, in the neighborhood of the point $(x_0, y_1^0, \cdots, y_n^0)$, and let it be constant along a solution 4):

$$\varpi[x, \varphi_1, \cdots, \varphi_n] \equiv c.$$

Then the equation

$$\varpi(x, y_1, \cdots, y_n) = c$$

is said to be an *integral* of the system of differential equations.

If

$$\varpi_1(x, y_1, \cdots, y_n), \qquad \varpi_2(x, y_1, \cdots, y_n)$$

are two integrals, they are said to be *independent* if the matrix

$$\left\|\begin{matrix} \dfrac{\partial \varpi_1}{\partial y_1} & \cdots & \dfrac{\partial \varpi_1}{\partial y_n} \\[2ex] \dfrac{\partial \varpi_2}{\partial y_1} & \cdots & \dfrac{\partial \varpi_2}{\partial y_n} \end{matrix}\right\|$$

is of rank 2. — The generalization of the definition to the case of $k$ integrals is obvious.

## EXERCISES

1. Let

$$c_k = \omega_k(b_1, \cdots, b_n), \qquad k = 1, \cdots, n,$$

where $\omega_k(b_1, \cdots, b_n)$ is continuous, together with its derivatives of the first order, throughout the neighborhood of the point $(b_1, \cdots, b_n) = (y_1^0, \cdots, y_n^0)$, and the Jacobian

$$\frac{\partial(c_1, \cdots, c_n)}{\partial(b_1, \cdots, b_n)} \neq 0.$$

Let

$$\varphi_k(x; b_1, \cdots, b_n) = \psi_k(x; c_1, \cdots, c_n), \qquad k = 1, \cdots, n.$$

Show that the equations

$$y_k = \psi_k(x; c_1, \cdots, c_n), \qquad k = 1, \cdots, n,$$

express the *general solution* of the given system of differential equations 1), § 3, — in the *two-fold* sense in which that term has been explained.

State your final result accurately as an independent theorem.

§ **7. Linear Differential Equations.** THEOREM 1. *Consider the system of linear differential equations of the first order:*

1) $$\frac{dy_k}{dx} = P_{1k}y_1 + \cdots + P_{nk}y_n + P_k,$$

$k = 1, \cdots, n,$ *where* $P_{jk} = P_{jk}(x)$ *and* $P_k = P_k(x)$ *are continuous in a closed interval:*

2) $$a \leqq x \leqq b.$$

*Let* $b_1, \cdots, b_n$ *be an arbitrary set of numbers, and let* $x_0$ *be an arbitrary point of* 2). *Then there exists a solution of* 1)*:*

3) $$y_k = \varphi_k(x), \qquad k = 1, \cdots, n,$$

*where* $\varphi_k(x)$ *is continuous, together with its first derivative, in the closed interval* 2) *and*

$$\varphi_k(x_0) = b_k, \qquad k = 1, \cdots, n.$$

*Moreover, the solution is unique.*

The theorem comes under the more general one of § 3, Exercise 4. By means of this theorem it is possible to treat the linear differential equation of the $n$-th order:

4) $$\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_ny = p,$$

where the coefficients $p_k = p_k(x)$ and $p = p(x)$ are continuous in a closed interval

5) $$a \leqq x \leqq b.$$

For on setting

6) $$\frac{dy_k}{dx} = y_{k+1},$$

we have:

7) $$\begin{cases} \dfrac{dy}{dx} = y_1, \\ \dfrac{dy_1}{dx} = y_2, \\ \cdots\cdots, \\ \dfrac{dy_{n-2}}{dx} = y_{n-1}, \\ \dfrac{dy_{n-1}}{dx} = -p_1 y_{n-1} - \cdots - p_n y + p. \end{cases}$$

We are thus led to the following existence theorem.

THEOREM. 2. *The linear differential equation of the n-th order Equation* 4), *admits a solution defined throughout the interval* 5)*:*

$$y = \varphi(x), \qquad a \leqq x \leqq b,$$

*such that, if* $c_0, c_1, \cdots, c_{n-1}$ *be any n numbers whatever,*

$$\varphi(x_0) = c_0, \quad \varphi'(x_0) = c_1, \quad \cdots, \quad \varphi^{(n-1)}(x_0) = c_{n-1}.$$

*The solution is unique.*

**§8. Differential Equations of Higher Order; General Case.** The general differential equation of the $n$-th order can be written in the form:

1) $$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right),$$

where $f(x, y, y_1, \cdots, y_{n-1})$ is continuous, together with its first derivatives with respect to $y, y_1, \cdots, y_{n-1}$, in the neighborhood of a point $(x_0, y_0, y_1^0, \cdots, y_{n-1}^0)$. On making the substitution 6) of § 7, the given differential equation 1) is replaced by the system 7) of § 7, except that the last equation 7) now becomes:

2) $$\frac{dy_{n-1}}{dx} = f(x, y, y_1, \cdots, y_{n-1}).$$

The existence theorem is precisely similar to Theorem 2, § 7, except that the domain of definition of the solution is restricted to the neighborhood of the point $x = x_0$.

### EXERCISES

1. Extend the existence theorem to the most general case. Begin with two differential equations:

$$\frac{d^n y}{dx^n} = f(x, y, y', \cdots, y^{(n-1)}, z, z', \cdots, z^{(m-1)}),$$

$$\frac{d^m z}{dx^m} = \varphi(x, y, y', \cdots, y^{(n-1)}, z, z', \cdots, z^{(m-1)}).$$

2. Apply the result to the system:

$$\frac{d^2 q_k}{dt^2} + n_k^2 q_k = 0, \qquad k = 1, \cdots, m,$$

where $n_k$ is a positive constant.

**§ 9. Complex Variables.** Corresponding theorems hold in the domain of complex variables, but the proofs are simpler. Beginning with the simplest case,

$$\frac{dy}{dx} = f(x, y),$$

let $f(x, y)$ be a function of the two complex variables $x$ and $y$, analytic in the point $(x_0, y_0)$. Then there exists a solution,

$$y = \varphi(x),$$

analytic in the point $x = x_0$ and taking on the value $y_0$ there:

$$y_0 = \varphi(x_0).$$

Moreover, this solution is unique.

The proof can be given by the method of successive approximations, as before; but the details are simpler. For when once the series

$$b + (y_1 - y_0) + (y_2 - y_1) + \cdots$$

has been shown to be uniformly convergent throughout a certain two-dimensional, or *complex*, neighborhood of the point $x = x_0$, it follows that the series represents an analytic function and can, by a general theorem due to Weierstrass, be differentiated term-by-term. Moreover, the case that $f$ depends on a parameter, $\alpha$, the function $f(x, y, \alpha)$ of the three complex variables being analytic in the point $(x_0, y_0, \alpha_0)$, is dealt with immediately, without any additional analysis. The function $\varphi(x, \alpha)$ of the complex variables is analytic in the point $(x_0, \alpha_0)$, and so can be differentiated with respect to $\alpha$ an unlimited number of times, the series being differentiable term-by-term.

The generalization to the case of a simultaneous system of $n$ differential equations presents no difficulty.

**§10. Linear Partial Differential Equations of the First Order.** Consider the linear partial differential equation:

$$A) \qquad X_1 \frac{\partial u}{\partial x_1} + \cdots + X_n \frac{\partial u}{\partial x_n} = 0,$$

where $X_k = X_k(x_1, \cdots, x_n)$ is continuous, together with its first derivatives, in the neighborhood of a point $(a_1, \cdots, a_n)$, and not all the $X_k$ vanish there.

Consider, secondly, the simultaneous system of total differential equations:

$$B) \qquad \frac{dx_1}{X_1} = \cdots = \frac{dx_n}{X_n}.$$

To solve Equation $A)$ is equivalent to solving equations $B)$; and conversely.

The theorem is adequately illustrated in the case $n = 3$:

$$A') \qquad X \frac{\partial u}{\partial x} + Y \frac{\partial u}{\partial y} + Z \frac{\partial u}{\partial z} = 0;$$

$$B') \qquad \frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}.$$

The solution of $B')$ is given by the equations:

$$1) \qquad y = f(x;\, x_0, y_0, z_0), \qquad z = \varphi(x;\, x_0, y_0, z_0),$$

provided $X(a, b, c) \neq 0$, where $(x_0, y_0, z_0)$ lies in the neighborhood of $(a, b, c)$. Let $x_0 = a$ be held fast, and interpret $(y_0, z_0)$ as a point in the plane $x = a$, lying near $(y, z) = (b, c)$. Then the equations

2) $$y=f(x;\, a, y_0, z_0), \qquad z=\varphi(x;\, a, y_0, z_0)$$

represent a two-parameter family (a so-called *congruence*) of curves coursing the neighborhood of $(a, b, c)$. No two of these curves intersect, for then there would be two solutions of $B')$ through such a point. Moreover, through each point of the neighborhood passes a curve 2). For, the implicit form of 1) is

3) $$y_0=f(x_0;\, x, y, z), \qquad z_0=\varphi(x_0;\, x, y, z).$$

Hence 2) is equivalent to

4) $$y_0=f(a;\, x, y, z), \qquad z_0=\varphi(a;\, x, y, z).$$

We see, then, that an arbitrary point $(x_1, y_1, z_1)$ near $(a, b, c)$ will lead through 4) to a point $(y_0, z_0)$ in the plane $x=a$, near $(b, c)$; and the curve 2) corresponding to these values of $y_0$, $z_0$ will go through $(x_1, y_1, z_1)$.

*Integral Surfaces.* Consider the surface:

5) $$f(a;\, x, y, z)=\text{const.} \qquad (=y_0).$$

It cuts the plane $x=a$ in the line $y=y_0=$ const. For,

6) $$f(x_0;\, x_0, y_0, z_0)\equiv y_0$$

is an identity in $(x_0, y_0, z_0)$. Hence

$$f(a;\, a, y, z)\equiv y,$$

no matter how $(y, z)$ be chosen near $(b, c)$. Because of 5), $y$ must have the value $y_0$; but $z$ is arbitrary.

The equation 5):

$$f(a;\, x, y, z)=y_0$$

can be solved for $y$; for

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0, z_0)}=1.$$

Hence 5) can be represented in the form:

7) $$y=\chi(x, z).$$

The surface 5) is an integral surface of the differential equations $B')$, as has already been pointed out in § 6. In particular, then, the solution of $B'$) which passes through a point $(x_1, y_1, z)$ of 5), cuts the plane $x = a$ in the line $y = y_0$. Thus the surface 5) or 7) is swept out by the solutions of $B'$) which pass through the points of the line $y = y_0$ in the plane $x = a$, near $z = c$.

Similar remarks apply to the surface

8) $$\varphi(a, x, y, z) = \text{const.} \qquad (= z_0).$$

This surface cuts the plane $x = a$ in the line $z = z_0 = \text{const.}$ Equation 8) can be solved for $z$:

9) $$z = - (x, y).$$

This surface is swept out by the solutions of $B'$) which pass through points of the line $z = z_0$ in the plane $x = a$, near $y = b$.

Consider now more generally an arbitrary regular curve of the plane $x = a$, which passes through $(y, z) = (b, c)$:

10) $$\Omega(y_0, z_0) = 0,$$

where the first derivatives, $\Omega_1$ and $\Omega_2$, exist and are continuous near $(b, c)$, and not both vanish there. The equation

11) $$\Omega(f, \varphi) = 0$$

represents a surface, swept out by the solutions of $B'$) which cut the plane $x = a$ in a point $(y_0, z_0)$ of the curve 10). For the points that lie on these solutions obviously are points of 11); and conversely, a point $(x_1, y_1, z_1)$ of 11) near $(a, b, c)$ determines a pair of values $y_0, z_0$:

$$y_0 = f(a;\, x_1, y_1, z_1), \qquad z_0 = \varphi(a;\, x_1, y_1, z_1)$$

which satisfy 10).

*Solution of the Partial Differential Equation $A'$).* The function

12) $$u = f(a, x, y, z)$$

is a solution of $A'$). For, the normal to the surface 5) in an arbitrary point is perpendicular to the solution of $B'$) through this point. Hence

13) $$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0.$$

Similarly, the function

$$u = \varphi(a;\ x, y, z)$$

is a solution of $A'$) and we have

$$X\frac{\partial \varphi}{\partial x} + Y\frac{\partial \varphi}{\partial y} + Z\frac{\partial \varphi}{\partial z} = 0. \tag{14)}$$

More generally,

$$u = \Omega(f, \varphi) \tag{15)}$$

is a solution. For

$$\begin{cases} \dfrac{\partial u}{\partial x} = \Omega_1\dfrac{\partial f}{\partial x} + \Omega_2\dfrac{\partial \varphi}{\partial x}, \\ \dfrac{\partial u}{\partial y} = \Omega_1\dfrac{\partial f}{\partial y} + \Omega_2\dfrac{\partial \varphi}{\partial y}, \\ \dfrac{\partial u}{\partial z} = \Omega_1\dfrac{\partial f}{\partial z} + \Omega_2\dfrac{\partial \varphi}{\partial z}. \end{cases}$$

On multiplying by $X$, $Y$, $Z$ respectively and adding, the result follows from 13) and 14).

Finally, let

$$u = \Psi(x, y, z) \tag{16)}$$

be an arbitrary solution of $A'$), where $\Psi$ is continuous together with its first partial derivatives in the neighborhood of the point $(a, b, c)$ and not all the derivatives vanish there. The surface

$$\Psi(x, y, z) = \Psi(a, b, c) \tag{17)}$$

is tangent at an arbitrary one of its points to the solution of $B'$) which passes through this point, as is clear from $A'$). In particular, then, the surface is not tangent to the plane $x = a$.

Moreover, the solution of $B'$) which goes through a point of 17), lies wholly in the surface. For, when $y$ and $z$ are given by 1),

$$\frac{d\Psi}{dx} = \Psi_x + \Psi_y\frac{Y}{X} + \Psi_z\frac{Z}{X} = 0.$$

It follows, then, that the surface 17) is swept out by the solutions of $B'$) which pass through the intersection of the surface 17) with the plane $x = a$.

There is still one point to be considered. The surface 17) cuts the plane $x = a$ in a curve 10), for since this surface is not tangent to the plane $x = a$, it follows that not both the derivatives $\partial \Psi / \partial y$ and $\partial \Psi / \partial z$ can vanish. The surface 17), therefore, is swept out by those solutions of $B'$) which pass through the points of this curve. The converse, however, requires proof, for it is not evident that the solutions of $B'$) which pass through the points of 10) sweep out a surface 17). Let

$$\Omega (f, \varphi) \equiv F(x, y, z),$$

where $\Omega (y_0, z_0)$ is *any* function 10). Then we will show that $F_y$, $F_z$ are not both 0. We have:

$$F_y = \Omega_1 \frac{\partial f}{\partial y} + \Omega_2 \frac{\partial \varphi}{\partial y},$$

$$F_z = \Omega_1 \frac{\partial f}{\partial z} + \Omega_2 \frac{\partial \varphi}{\partial z}.$$

Now,

$$\frac{\partial (f, \varphi)}{\partial (y, z)} \neq 0,$$

and $\Omega_1$, $\Omega_2$ are not both 0. Hence $F_y$, $F_z$ cannot vanish simultaneously, q. e. d.

The result may be stated in a somewhat more general form as follows.

SOLUTION OF $A'$) BY $B'$). *Let $\Gamma$ be a regular curve drawn through the point $(a, b, c)$, not tangent to the solution of $B'$) there. The solutions of $B'$) which pass through the points of $\Gamma$ sweep out a surface,*

20) $$\Psi (x, y, z) = C,$$

*where $\Psi$ is continuous together with its first partial derivatives in the neighborhood of $(a, b, c)$, and not all of the latter vanish. The function*

21) $$u = \Psi (x, y, z)$$

*is a solution of $A'$). And conversely, any solution 21) of $A'$), such that $\Psi$ satisfies the above conditions of continuity, is obtainable in this manner.*

*Solution of the Total Differential Equations $B'$).* We proceed now to solve the system of total differential equations $B'$) by means of the single partial differential equation $A'$). Let

$$22)\qquad u = \Psi(x, y, z), \qquad u = \Theta(x, y, z)$$

be two solutions of $A'$) such that the rank of the matrix

$$23)\qquad \left\| \begin{matrix} \dfrac{\partial \Psi}{\partial x} & \dfrac{\partial \Psi}{\partial y} & \dfrac{\partial \Psi}{\partial z} \\[2ex] \dfrac{\partial \Theta}{\partial x} & \dfrac{\partial \Theta}{\partial y} & \dfrac{\partial \Theta}{\partial z} \end{matrix} \right\|$$

in the point $(a, b, c)$ is 2. Then the curve

$$24)\qquad \Psi(x, y, z) = \alpha, \qquad \Theta(x, y, z) = \beta,$$

where $(\alpha, \beta)$ is a point of the neighborhood of $(\alpha_0, \beta_0)$:

$$\Psi(a, b, c) = \alpha_0, \qquad \Theta(a, b, c) = \beta_0,$$

is a solution of $B'$). For each of the surfaces 24) contains the solution of $B'$) which passes through a point of intersection of these surfaces. — We may state the result as follows.

SOLUTION OF $B'$) BY $A'$). *Let $\Psi(x, y, z)$ and $\Theta(x, y, z)$ be two solutions of $A'$) satisfying the conditions that the matrix* 23) *is of rank* 2. *Then the solutions of $B'$) are given by the equations* 24).

**§11. The General Case.** It is now easy to state the theorem in the general case. Given the linear partial differential equation $A$) of § 10 and the system of $n$ total differential equations $B$), let at least one $X_k$ be different from 0:

$$X_1(a_1, \cdots, a_n) \neq 0.$$

Through each point $(x^0) = (x_1^0, \cdots, x_n^0)$ of the neighborhood of $(a)$ passes one and only one solution of $B$), represented by the equations

$$x_i^0 = \varphi_i(a_1; x_1, \cdots, x_n), \qquad i = 2, \cdots, n;$$

$$\frac{\partial(\varphi_2, \cdots, \varphi_n)}{\partial(x_2, \cdots, x_n)} \neq 0.$$

In particular, there is a one-to-one relation between these solutions and the points $(a_1, x_2^0, \cdots, x_n^0)$ in which they cut the plane $x_1 = a_1$.

Each function

$$u = \varphi_i(a_1;\, x_1, \cdots, x_n)$$

is a solution of $A$). More generally, let

$$\Omega\,(x_2, \cdots, x_n)$$

be continuous, together with its first derivatives, at least one of which shall not vanish in $(a_2, \cdots, a_n)$, and let

$$\Omega\,(a_2, \cdots, a_n) = 0.$$

Then the function

$$u = \Omega\,(\varphi_2, \cdots, \varphi_n)$$

is also a solution of $A$). It is, moreover, the most general solution,

$$u = \Psi\,(x_1, \cdots, x_n),$$

which has the property that at least one first derivative (here, $\partial u/\partial x_1$) does not vanish. Thus $A$) is completely solved by $B$).

Conversely, $B$) is completely solved by $n - 1$ solutions of $A$):

$$u = \Psi_1\,(x_1, \cdots, x_n), \quad \cdots, \quad \Psi_{n-1}\,(x_1, \cdots, x_n),$$

considered in the neighborhood of the point $(x) = (a)$, provided that the rank of the matrix

$$\left\Vert \begin{matrix} \dfrac{\partial \Psi_1}{\partial x_1} & \cdots\cdots & \dfrac{\partial \Psi_1}{\partial x_n} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \dfrac{\partial \Psi_{n-1}}{\partial x_1} & \cdots\cdots & \dfrac{\partial \Psi_{n-1}}{\partial x_n} \end{matrix} \right\Vert$$

in the point $(a)$ is $n - 1$. The $n - 1$ equations:

$$\Psi_i\,(x_1, \cdots, x_n) = C_i, \qquad i = 2, \cdots, n,$$

determine the solutions of $B$).

**§12. Change of Variables.** Let

$$1) \qquad y_k = \varpi_k(x_1, \cdots, x_n), \qquad k = 1, \cdots, n,$$

where $\varpi_k(x_1, \cdots, x_n)$ has continuous first derivatives in the neighborhood of the point $(x) = (a)$ and

$$2) \qquad \frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)} \neq 0$$

in $(x) = (a)$. Then $A)$ and $B)$ go over into

$$Z_1 \frac{\partial u}{\partial y_1} + \cdots + Z_n \frac{\partial u}{\partial y_n} = 0,$$

$$\frac{dy_1}{Z_1} = \cdots = \frac{dy_n}{Z_n}.$$

Let

$$Y_k = \rho Z_k, \qquad k = 1, \cdots, n,$$

where $\rho$ is any non-vanishing function of $(x_1, \cdots, x_n)$, continuous together with its first derivatives near $(a)$. Then the solutions of $A)$ and $B)$ will go over into the solutions of

$$Y_1 \frac{\partial u}{\partial y_1} + \cdots + Y_n \frac{\partial u}{\partial y_n} = 0;$$

$$\frac{dy_1}{Y_1} = \cdots = \frac{dy_n}{Y_n}.$$

It appears, then, that the equations $A)$ and $B)$ are invariant of a transformation 1). We can use this fact to reduce them to a simpler system, in case one or more integrals are known. Suppose that $m < n$ independent integrals are known,

$$\Psi_1(x_1, \cdots, x_n), \quad \cdots, \quad \Psi_m(x_1, \cdots, x_n),$$

where the matrix

$$3) \qquad \left\| \begin{matrix} \frac{\partial \Psi_1}{\partial x_1} \cdots\cdots \frac{\partial \Psi_1}{\partial x_n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \frac{\partial \Psi_m}{\partial x_1} \cdots\cdots \frac{\partial \Psi_m}{\partial x_n} \end{matrix} \right\|$$

is of rank $m$ in the point $(a_1, \cdots, a_n)$. Make a change of variables, setting (in case the determinant corresponding to the first $m$ columns of the matrix is $\neq 0$)

$$4)\qquad y_k = \Psi_k(x_1, \cdots, x_n), \qquad k = 1, \cdots, m,$$

with the further $y$'s any functions such that 2) is satisfied. The transformed equations admit the integrals

$$y_1, \cdots, y_m,$$

and hence

$$Y_1 = 0, \quad \cdots, \quad Y_m = 0.$$

The transformed system thus takes the form, if $m < n - 1$:

$$5)\qquad \begin{cases} Y_{m+1} \dfrac{\partial u}{\partial y_{m+1}} + \cdots + Y_n \dfrac{\partial u}{\partial y_n} = 0, \\[2ex] \dfrac{dy_{m+1}}{Y_{m+1}} = \cdots = \dfrac{dy_n}{Y_n}. \end{cases}$$

These equations can be solved by setting

$$6)\qquad y_k = c_k, \qquad k = 1, \cdots, m,$$

where the $c_k$ are suitably restricted arbitrary constants, and then considering the new system. We have thus been led to a system of equations like the original system, the number of variables, however, having been reduced from $n$ to $n - m$.

Transforming back to the original equations in the $x$'s, i.e. to Equations *A*) and *B*), we see that $x_1, \cdots, x_m$ can be determined from the equations

$$7)\qquad \Psi_1(x_1, \cdots, x_n) = c_1, \quad \cdots, \quad \Psi_m(x_1. \cdots, x_n) = c_m.$$

Thus the equations

$$8)\qquad \frac{dx_{m+1}}{X_{m+1}} = \cdots = \frac{dx_n}{X_n}$$

yield, on replacing $x_1, \cdots, x_m$ in the $X$'s by their values from 7), a system in which only $n - m$ variables, $x_{m+1}, \cdots, x_n$ appear.

The justification for the elimination in both cases lies in the theorem of uniqueness. *One* solution of Equations 5) is given by 6) and the $n - m$ further equations. But there is *only one* solution.

One question, however, still remains open. We started with the hypothesis that at least one of the $X$'s did not vanish in the point $(a_1, \cdots, a_n)$. Are we sure that there will be a $Y$ which will not vanish? Otherwise we could not apply the existence theorem to Equations 5). From 4) we have:

$$dy_k = c_{1k}\, dx_1 + \cdots + c_{nk}\, dx_n, \qquad k = 1, \cdots, n,$$

where the determinant of these equations is the Jacobian 2). From

$$\frac{dx_1}{X_1} = \cdots = \frac{dx_n}{X_n}$$

follows that each of these ratios has the value:

$$\frac{c_{1k}\, dx_1 + \cdots + c_{nk}\, dx_n}{c_{1k} X_1 + \cdots + c_{nk} X_n}.$$

Hence these denominators cannot all vanish, and so, in particular, at least one of the $Y_{m+1}, \cdots. Y_n$ must be different from 0.

## §13. The General Partial Differential Equation of the First Order; $n = 2$.

Consider the partial differential equation

1) $$F(x, y, z, p, q) = 0,$$

where

2) $$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}.$$

Here, $F(x, y, z, p, q)$, together with its partial derivatives of the first two orders, shall be continuous in a region of the space of the $(x, y, z, p, q)$, where $(x, y, z)$ is any point of a region $R$ which includes none of its boundary points, and $p, q$ are wholly unrestricted*. Finally, $\partial F/\partial p$ and $\partial F/\partial q$ shall not vanish simultaneously.

What does the differential equation 1) mean geometrically? We seek an answer analogous to that given in § 1 for ordinary differential equations.

Let $(x_0, y_0, z_0)$ be a point of $R$, and let

3) $$z = \Phi(x, y)$$

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* More generally, we may impose the requirement for the neighborhood of a point $(a, b, c, \alpha, \beta)$. The treatment is essentially the same.

be a solution of 1), the surface $S$ represented by 3) passing through $(x_0, y_0, z_0)$ It is assumed that $\Phi(x, y)$, together with its first derivatives, is continuous in the neighborhood of $(x_0, y_0)$. The direction components of the normal to $S$ are $(p_0, q_0, -1)$, where

$$p_0 = \Phi_x(x_0, y_0), \qquad q_0 = \Phi_y(x_0, y_0).$$

They are subject to the condition:

$$4) \qquad F(x_0, y_0, z_0, p_0, q_0) = 0.$$

This fact gives us the clue to the geometric interpretation of 1) which we are looking for. Let $(x, y, z)$ be an arbitrary point of $R$, and hold it fast. Then 1) defines a one-parameter family of lines through $(x, y, z)$, namely, those whose direction-components are $(p, q, -1)$; and these lines generate a cone, $N$, — the cone of normals to the solutions of 1) which pass through $(x, y, z)$.

The planes through $(x, y, z)$ normal to these lines envelop* a cone $T$. These planes are the tangent planes to the solutions of 1) which pass through $(x, y, z)$, and each generator of the cone lies in one of these planes. We thus have, assigned to each point of $R$, a *cone* $T$, and any solution of 1) defines a surface $S$ in $R$ which is tangent at each of its points to the cone corresponding to this point.

This is hardly enough, however, to enable us to determine the integral surfaces $S$ of 1). But we can go a step further. Turn back to the point $(x_0, y_0, z_0)$ and consider the tangent planes pertaining to this point. They are given by the equation

$$5) \qquad z - z_0 = p(x - x_0) + q(y - y_0),$$

where $p$, $q$ are parameters connected by the equation:

$$6) \qquad F(x_0, y_0, z_0, p, q) = 0.$$

Their envelope is determined by 5) and the further equation:

$$7) \qquad 0 = \frac{\partial p}{\partial q}(x - x_0) + (y - y_0),$$

in case $\partial F/\partial p \neq 0$, the condition mentioned in the foot-note being fulfilled. Introduce the notation:

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* A further condition of inequality here is needed; cf. the Author's *Advanced Calculus*, p. 194 and p. 564.

8) $$X = \frac{\partial F}{\partial x}, \quad Y = \frac{\partial F}{\partial y}, \quad Z = \frac{\partial F}{\partial z}, \quad P = \frac{\partial F}{\partial p}, \quad Q = \frac{\partial F}{\partial q}.$$

Then 7) is equivalent to

9) $$\frac{x - x_0}{P} = \frac{y - y_0}{Q}.$$

We now have new geometrical data relating to the surface 3). It not only is tangent to the cone $T$ in $(x_0, y_0, z_0)$, but we can assign a tangent *line* to $S$ in this point, namely, that generator of $T$ which is tangent to $S$. Thus to each point of $S$ is assigned a *direction* lying in the surface and having the direction components $dx, dy, dz$ satisfying the relations:

10) $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{pP + qQ}.$$

More precisely, we think of a curve drawn through $(x_0, y_0, z_0)$ on the surface $S$:

11) $$x = f(\lambda), \qquad y = \varphi(\lambda), \qquad z = \psi(\lambda),$$

having in $(x_0, y_0, z_0)$ its direction components given by the equations:

12) $$\frac{dx}{d\lambda} = P, \qquad \frac{dy}{d\lambda} = Q, \qquad \frac{dz}{d\lambda} = pP + qQ.$$

We see, then, that the surface $S$ is swept out by a one-parameter family of such curves. They are determined through the existence theorem of § 3 (including, in particular, the property of uniqueness) by the system of differential equations:

$$\frac{dy}{dx} = \frac{Q}{P}, \qquad \frac{dz}{dx} = \frac{pP + qQ}{P},$$

in case $P \neq 0$; otherwise by

$$\frac{dx}{dy} = \frac{P}{Q}, \qquad \frac{dz}{dy} = \frac{pP + qQ}{Q},$$

where $p$, $q$ are given by 3).

Let us follow one of these curves along the surface $S$. The curve can be represented by Equations 11), thought of as an integral of 12), $p$ and $q$ being obtained from 3). Now, turn to Equation 1). This becomes an identity in $x$ and $y$ when we substitute for $z, p, q$

their values from 3) and 2). Hence the partial derivatives with respect to $x$ and $y$ vanish identically, or

$$13)\quad \begin{cases} X + pZ + P\dfrac{\partial p}{\partial x} + Q\dfrac{\partial q}{\partial x} = 0 \\[2ex] Y + qZ + P\dfrac{\partial p}{\partial y} + Q\dfrac{\partial q}{\partial y} = 0 \end{cases}$$

On the other hand, along the curve 11), we have from 12):

$$14)\quad \begin{cases} \dfrac{dp}{d\lambda} = \dfrac{\partial p}{\partial x}\dfrac{dx}{d\lambda} + \dfrac{\partial p}{\partial y}\dfrac{dy}{d\lambda} = P\dfrac{\partial p}{\partial x} + Q\dfrac{\partial q}{\partial x} \\[2ex] \dfrac{dq}{d\lambda} = \dfrac{\partial q}{\partial x}\dfrac{dx}{d\lambda} + \dfrac{\partial q}{\partial y}\dfrac{dy}{d\lambda} = P\dfrac{\partial p}{\partial y} + Q\dfrac{\partial q}{\partial y} \end{cases}$$

Hence

$$15)\quad \frac{dp}{d\lambda} = -X - pZ, \qquad \frac{dq}{d\lambda} = -Y - qZ.$$

We may write the final result in the form:

$$16)\quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{pP + qQ} = \frac{-dp}{X + pZ} = \frac{-dq}{Y + qZ}.$$

These equations, irrespective of any particular integral surface 3), and also irrespective of the condition of inequality, foot-note, p. 388, define a four-parameter family of curves in the five-dimensional space of the variables $(x, y, z, p, q)$. Since $P$ and $Q$ are not both 0 in the point $A$: $(a, b, c, \alpha, \beta)$, let $P \neq 0$ there. The primitive of 16) can then be written in the form:

$$17)\quad \begin{cases} y = f_1(x; x_0, y_0, z_0, p_0, q_0) \\ z = f_2(x; x_0, y_0, z_0, p_0, q_0) \\ p = f_3(x; x_0, y_0, z_0, p_0, q_0) \\ q = f_4(x; x_0, y_0, z_0, p_0, q_0) \end{cases}$$

Each function $f_k$ admits a continuous first partial derivative with respect to each of the six arguments, $x; x_0, \cdots$; § 5. The partial derivative $\partial f_1/\partial x$ is the same as the total derivative $dy/dx$ in 16), or

$$18)\quad \frac{\partial f_1}{\partial x} = \frac{dy}{dx} = \frac{Q}{P};$$

and similarly for each of the other $\partial f_k/\partial x$. Moreover, the order of differentiation between $x$ and any one of the arguments $x_0, y_0, z_0, p_0, q_0$ can always be reversed, § 5:

$$19)\qquad \frac{\partial}{\partial y_0}\frac{dy}{dx} = \frac{\partial^2 y}{\partial x\,\partial y_0}, \qquad \text{etc.}$$

*The Curves C.* The four-parameter family of curves 17) shall be denoted as the *Curves C.* They sweep out the neighborhood of the point $A$ just once. Without loss of generality we can set $x_0 = a$, and then $y_0, z_0, p_0, q_0$ represent the four independent parameters of the family, the equations now taking the form:

$$20)\qquad \begin{cases} y = f_1(x\,;\, a, y_0, z_0, p_0, q_0) \\ z = f_2(x\,;\, a, y_0, z_0, p_0, q_0) \\ p = f_3(x\,;\, a, y_0, z_0, p_0, q_0) \\ q = f_4(x\,;\, a, y_0, z_0, p_0, q_0) \end{cases}$$

Along any one of the curves $C$ the function $F$ is constant:

$$21)\qquad F(x, y, z, p, q) = C,$$

as appears at once from 16):

$$22)\qquad \frac{dF}{dx} = X + Y\frac{dy}{dx} + Z\frac{dz}{dx} + P\frac{dp}{dx} + Q\frac{dq}{dx} = 0.$$

Thus we have in 21) an integral of the simultaneous system of total differential equations 16).

From the foregoing analysis and the theorem of uniqueness we can infer the following theorem.

THEOREM 1. *Let* $(x_0, y_0, z_0, p_0, q_0)$ *be a point of the neighborhood of* $A$, *such that*

$$F(x_0, y_0, z_0, p_0, q_0) = 0.$$

*Let*

$$23)\qquad z = \Psi(x, y)$$

*be an integral of the partial differential equation* 1), *where* $\Psi(x, y)$ *is continuous, together with its first derivatives, in the neighborhood of* $(x, y) = (x_0, y_0)$, *and where, furthermore,*

$$\Psi(x_0, y_0) = z_0, \qquad \Psi_x(x_0, y_0) = p_0, \qquad \Psi_y(x_0, y_0) = q_0.$$

*Then the curve defined by the first two equations* 17) *lies wholly in the surface represented by* 23), *and at each point of this curve*

$$\Psi_x(x, y) = p, \qquad \Psi_y(x, y) = q$$

*where* $p$, $q$ *are given by the last two equations* 17).

*Parametric Form.* The curves $C$ can be represented in parametric form as follows. Replace 16) by the extended system:

$$24)\quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{pP + qQ} = \frac{-dp}{X + pZ} = \frac{-dq}{Y + qZ} = \frac{du}{G},$$

where $G = G(x, y, z, p, q, u)$ is continuous, together with its second derivatives, in the neighborhood of the point $(a, b, c, \alpha, \beta, 0)$ and does not vanish there. If, in particular, $G = 1$, then $u$ is usually represented by $t$:

$$\frac{du}{G} = dt.$$

Or, again, we may set $G = P$, when $P \neq 0$, and then $u = x$, and we fall back on the integral in the form 17) or 20).

In the general case we have as the integral of 24):

$$25)\quad \begin{cases} x = \varphi_1(u;\, x_0, y_0, z_0, p_0, q_0) \\ y = \varphi_2(u;\, x_0, y_0, z_0, p_0, q_0) \\ z = \varphi_3(u;\, x_0, y_0, z_0, p_0, q_0) \\ p = \varphi_4(u;\, x_0, y_0, z_0, p_0, q_0) \\ q = \varphi_5(u;\, x_0, y_0, z_0, p_0, q_0) \end{cases}$$

These equations, for values of $u$ near 0, represent the curves C.

*Characteristics.* The curves $C$ form the basis for the definition of the *characteristics* of the partial differential equation 1). Sometimes these curves themselves are called characteristics. More narrowly, it is the curves $C$ for which

$$26)\quad F(x_0, y_0, z_0, p_0, q_0) = 0$$

that have especial importance for the integration of 1). The first two of equations 17) or the first three of equations 25) define a curve in the $(x, y, z)$-space and assign to each point of this curve a tangent plane, the direction components of whose normal, $(p, q, -1)$, are given by the last two equations. Such a curve and the tangent planes associated with its points are called a *characteristic strip*. A characteristic strip is said to *lie in a surface* if the curve lies in the surface and each associated plane is tangent to the surface.

In the foregoing developments is contained the proof of the following theorem.

THEOREM 2. *If $S$ is the surface corresponding to a solution of* 1) *and if the curve of a characteristic strip meets $S$ in a point $O$, the associated plane at $O$ being tangent to $S$, then the whole characteristic strip lies in $S$.*

*It is understood that $S$ is a regular surface such as is given by* 23).

There is a three-parameter family of characteristic strips, and it can be represented by 20), subject to the condition

$$27)\qquad F(a, y_0, z_0, p_0, q_0) = 0.$$

The curves of these strips, represented by the first two equations 20), combined with 27), course the neighborhood of the point $(x, y, z) = (a, b, c)$ and carry with them their tangent planes. The problem of integrating 1) is to fit into this neighborhood, in all possible ways, surfaces which coincide at each point with a characteristic strip. We turn now to the solution of this problem.

**§ 14. Continuation. Integration by Means of Characteristics.** Let a curve

$$28)\qquad z_0 = \omega(y_0)$$

be drawn in the plane $x = a$. Here, $\omega(y)$ shall be continuous, together with its first and second derivatives, in the neighborhood of the point $y = b$, and moreover $\omega(b)$ shall $= c$:

$$c = \omega(b).$$

Restrict $y_0$ to the neighborhood of $b$.

Through each point of this curve passes a curve $C$, defined as follows. $y_0$ and $z_0$ shall be connected by 28). Furthermore, $q_0$ shall have the value:

$$q_0 = \frac{dz_0}{dy_0} = \omega'(y_0).$$

Finally, $p_0$ shall be given by 27). Thus $y_0$ appears as a parameter; set it equal to $v$:

$$y_0 = v.$$

These curves $C$ sweep out a surface, defined parametrically by the first three equations 25), where we shall find it convenient to take the parameter $u$ as $x$, and to set $x_0 = a$. Thus $\varphi_1 \equiv u$ and equations 25) reduce essentially to 20), but the exposition is clearer in the parametric form:

29) $$x = f(u, v), \qquad y = \varphi(u, v), \qquad z = \psi(u, v).$$

First of all,

$$J = \frac{\partial(x, y)}{\partial(u, v)} \neq 0.$$

For, since $x = u$,

$$\frac{\partial x}{\partial u} = 1, \qquad \frac{\partial x}{\partial v} = 0.$$

Next,

$$\varphi_2(a; a, y_0; z_0, p_0, q_0) = y_0 = v,$$

and hence

$$\frac{\partial y}{\partial v} = 1 \qquad \text{when } u = a.$$

Thus $J = 1$ when $u = a$, $v = b$. It is, therefore, possible to solve the first two equations 29) for $u$ and $v$. Substituting these values in the third equation, we find:

30) $$z = \Phi(x, y),$$

where $\Phi(x, y)$, together with its first derivatives, is continuous in the neighborhood of $(a, b)$.

THEOREM 1. *The function* $\Phi(x, y)$ *is a solution of the given partial differential equation:*

31) $$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0.$$

The direction components of the normal to the surface 30) are $(\Phi_x, \Phi_y, -1)$. We have to show that $\Phi_x$, $\Phi_y$ make the equation

$$F(x, y, z, \Phi_x, \Phi_y) = 0$$

an identity in $(x, y)$. Now, from 30),

$$dz = \Phi_x\, dx + \Phi_y\, dy,$$

no matter what the independent variables are. And conversely, if

$$dz = p\, dx + q\, dy, \tag{32}$$

no matter what the independent variables may be, it follows that

$$p = \Phi_x, \qquad q = \Phi_y.$$

A necessary and sufficient condition for the truth of 32) is:

$$\begin{cases} \dfrac{\partial z}{\partial u} = p\dfrac{\partial x}{\partial u} + q\dfrac{\partial y}{\partial u}, \\[2ex] \dfrac{\partial z}{\partial v} = p\dfrac{\partial x}{\partial v} + q\dfrac{\partial y}{\partial v}. \end{cases} \tag{33}$$

The first equation is equivalent to

$$\frac{dz}{dx} = p + q\frac{dy}{dx},$$

and is true because of 16).

To establish the second equation, write

$$U(u, v) = \frac{\partial z}{\partial v} - p\frac{\partial x}{\partial v} - q\frac{\partial y}{\partial v}. \tag{34}$$

Along the curve 28)

$$\frac{\partial z}{\partial v} = \frac{\partial z_0}{\partial y_0} = \omega'(y_0), \quad \frac{\partial x}{\partial v} = \frac{\partial a}{\partial y_0} = 0, \quad \frac{\partial y}{\partial v} = \frac{\partial y_0}{\partial y_0} = 1,$$

and hence

$$U(a, v) = 0. \tag{35}$$

We will show that $U(u, v) = 0$ along each curve $C$.

To do this, compute $\partial U/\partial u$ along an arbitrary $C$. Remembering that, on account of 19), the order of differentiation can be reversed, we have:

$$\frac{\partial U}{\partial u} = \frac{\partial}{\partial v}\frac{\partial z}{\partial x} - \frac{dp}{dx}\frac{\partial x}{\partial v} - p\cdot 0 - \frac{dq}{dx}\frac{\partial y}{\partial v} - q\frac{\partial}{\partial v}\frac{dy}{dx} \tag{36}$$

$$= \frac{\partial}{\partial v}\frac{pP + qQ}{P} + \frac{X + pZ}{P}\frac{\partial x}{\partial v} + \frac{Y + qZ}{P}\frac{\partial y}{\partial v} - q\frac{\partial}{\partial v}\frac{Q}{P}$$

$$= \frac{1}{P}\left\{(X + pZ)\frac{\partial x}{\partial v} + (Y + qZ)\frac{\partial y}{\partial v} + P\frac{\partial p}{\partial v} + Q\frac{\partial q}{\partial v}\right\}.$$

On the other hand

37) $$F(x, y, z, p, q) = 0$$

along any curve $C$ and hence at every point of the surface 30); i.e. 37) is an identity in the variables $(u, v)$. Hence

$$0 = X\frac{\partial x}{\partial v} + Y\frac{\partial y}{\partial v} + Z\frac{\partial z}{\partial v} + P\frac{\partial p}{\partial v} + Q\frac{\partial q}{\partial v}.$$

On subtracting this equation, divided by $P$, from 36) and remembering 34) we find:

38) $$\frac{\partial U}{\partial u} = -\frac{Z}{P}U.$$

It is this equation which determines $U$ along a curve $C$. Regarding, then, $v$ as a parameter which characterizes $C$, we may write 38) in the form:

39) $$\frac{dU}{dx} = -\frac{Z}{P}U.$$

We seek the solution of this differential equation which vanishes when $x = a$. One such solution is $U = 0$. Because of the uniqueness theorem there is only one solution, and the proof is complete.

THEOREM 2. *Let*

40) $$z = \Psi(x, y)$$

*be a solution of* 1), *where* $\Psi(x, y)$ *is continuous, together with its derivatives of the first order, in the neighborhood of the point* $(a, b)$ *and* $\Psi(a, b) = c$. *Let* $\Gamma$ *be the curve in which the surface,* $S$, *represented by* 40) *is cut by the plane* $x = a$:

41) $$x = a, \qquad z = \Psi(x, y).$$

*Then the characteristic strips determined as above by* $\Gamma$ *sweep out* $S$ *just once.*

For, first, a characteristic strip determined by a point of $\Gamma$ and the tangent of $\Gamma$ at that point, lies wholly in $S$; § 13, Theorem 2

Secondly, at an arbitrary point of $S$ a characteristic strip is determined, which lies wholly in $S$. The curve of this strip cuts the plane $x = a$, and the point of intersection is necessarily a point of $\Gamma$.

This theorem is only partially a converse of Theorem 1. For in that theorem, the function $\omega(y)$ was required to possess a second derivative. But it can happen that the curve of Theorem 2 is such that $\Psi(a, y)$ has no second derivative; e g.

$$F(x, y, z, p, q) = p.$$

On the other hand, Theorem 1 is not true if $\omega(y)$ is required merely to possess a continuous first derivative, as is shown by the example:

$$F(x, y, z, p, q) = p + q^2,$$

$$\omega(u) = \frac{1}{2}\int_0^u f(u)\,du,$$

$$f(u) = u^{\frac{1}{2}} \sin\frac{1}{u}, \qquad u \neq 0; \qquad f(0) = 0.$$

Here, $(a, b, c) = (0, 0, 0)$.

The form which Theorem 1 should take, if it and Theorem 2 are to be each the converse of the other, is the following:—

THEOREM 1′: *Given the partial differential equation* 1). *Let* $\omega(y)$ *be continuous, together with its first derivative, in the neighborhood of the point* $y = b$, *and let* $\omega(b) = c$. *Let* $\Gamma$ *be the curve:*

$$x = a, \qquad z = \omega(y).$$

*Consider the characteristic strips determined by the points and tangents of* $\Gamma$. *If the curves of these characteristic strips sweep out a surface*

$$z = \Phi(x, y),$$

*where* $\Phi(x, y)$ *is single-valued and continuous, together with its derivatives of the first order, in the neighborhood of the point* $(a, b)$, *then* $\Phi(x, y)$ *will be a solution of* 1).

*A sufficient, but not a necessary, condition for the fulfillment of the last hypothesis is, that* $\omega(y)$ *possess a continous second derivative.*

*Uniqueness.* Observe that in any case two solutions of 1), which correspond to one and the same curve $\Gamma$, are identical.

More generally, let $S$ and $S'$ be the surfaces corresponding to two solutions of 1), and let $S$ and $S'$ be tangent to each other along a curve $\Gamma'$ which nowhere touches a characteristic curve of either surface Then the solutions are identical.

## § 15. The Case of $n$ Variables.

Both the theorem and the treatment by characteristics can be extended immediately to the general case. Let the partial differential equation

$$1) \qquad F\left(x_1, \cdots, x_n, z, \frac{\partial z}{\partial x_1}, \cdots, \frac{\partial z}{\partial x_n}\right) = 0$$

be given, where $F(x_1, \cdots, x_n, z, p_1, \cdots, p_n)$ is continuous, together with its partial derivatives of the first two orders, in the neighborhood of the point $A$: $(a_1, \cdots, a_n, c, b_1, \cdots, b_n)$. Write

$$X_k = \frac{\partial F}{\partial x_k}, \quad Z = \frac{\partial F}{\partial z}, \quad P_k = \frac{\partial F}{\partial p_k}.$$

Let the $P_k$ not all vanish in the point $A$.

The *characteristics* are defined on the hand of the $2n$ equations:

$$2) \qquad \frac{dx_k}{P_k} = \frac{dz}{\Sigma p_k P_k} = \frac{-dp_k}{X_k + p_k Z}, \qquad k = 1, \cdots, n.$$

These equations determine a $2n$-parameter family of curves $C$ in the $(2n+1)$-dimensional space of the variables, which, in case $P_1 \neq 0$ in $A$, can be represented in the form:

$$3) \qquad \begin{cases} x_k = f_k(x_1; x_1^0, \cdots, x_n^0, z^0, p_1^0, \cdots, p_n^0), & k = 2, \cdots, n; \\ z = f(x_1; x_1^0, \cdots, x_n^0, z^0, p_1^0, \cdots, p_n^0), & \\ p_k = f_{n+j}(x_1; x_1^0, \cdots, x_n^0, z^0, p_1^0, \cdots, p_n^0), & j = 1, \cdots, n. \end{cases}$$

Here, $x_1^0$ can be held fast:

$$4) \qquad x_1^0 = a_1,$$

and then $x_2^0, \cdots, x_n^0, z^0, p_1^0, \cdots, p_n^0$ afford a system of $2n$ independent parameters.

The *characteristic curves* are defined by the first $n$ of the equations 3), where the $(x^0, z^0, p^0)$ are subject to the conditions 4) and

$$5) \qquad F(x_1^0, \cdots, x_n^0, z^0, p_1^0, \cdots, p_n^0) = 0.$$

The *characteristic strips* are defined by all $2n$ of the equations 3) and are thought of as the characteristic curves, to each point of which is assigned a (hyper-) tangent plane, the direction-components of the normal being $(p_1^0, \cdots, p_n^0)$.

From here on the theory proceeds as in the earlier case, culminating in the

THEOREM. *Given the partial differential equation* 1) *with* $P_1 \neq 0$ *in* $A$. *Let* $\omega(x_2, \cdots, x_n)$ *be continuous, together with its partial derivatives of the first order, in the neighborhood of the point* $(a_2, \cdots, a_n)$ *and let*

$$\omega(a_2, \cdots, a_n) = c.$$

*Through each point of the manifold*

$$x = a_1, \qquad z^0 = \omega(x_2^0, \cdots, x_n^0)$$

*pass the characteristic strip determined by the point and the tangent plane; i. e set*

$$p_k^0 = \frac{\partial \omega}{\partial x_k^0}, \qquad k = 2, \cdots, n,$$

$p_1^0$ *being given by* 5).

*If the corresponding set of characteristic curves, determined by the first* $n$ *equations* 3), *sweep out a surface,*

$$z = \Phi(x_1, \cdots, x_n),$$

*where* $\Phi(x_1, \cdots, x_n)$ *is single-valued and, together with its partial derivatives of the first order, continuous in the neighborhood of* $(a_1, \cdots, a_n)$, *then* $\Phi(x_1, \cdots, x_n)$ *is a solution of* 1).

*A sufficient, but not a necessary, condition for the fulfillment of this last hypothesis is, that* $\omega(x_1, \cdots, x_n)$ *admit continuous derivatives of the second order.*

*Conversely, any solution of* 1) *satisfying the conditions imposed on* $\Phi(x_1, \cdots, x_n)$ *defines a surface which is swept out by characteristic curves as above.*

*Uniqueness.* The earlier theorem of uniqueness for the case $n = 2$ admits of direct generalization to the case $n = n$.